

The Quantum Passage

Quantum Mechanics from
the Structure of Comparison

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Preface

I. What This Book Derives

Quantum mechanics rests on six postulates: the Hilbert space of states, observables as self-adjoint operators, unitary evolution, the tensor product for composite systems, the update of the state on measurement, and the Born rule. The postulates were assembled from six traditions over about twenty-five years, and they were unified by their success. I derive all six from one source.

The source of all six postulates is the structure of comparison: what is necessarily involved in any act of relating one thing to another. The Immanent Metaphysics is a formal account of the structure of comparison. Chapter 3 presents the Immanent Metaphysics from its first step, and I assume no prior acquaintance with that account.

The derivation reaches past the general framework of quantum theory to the specific algebra of the actual world, $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and to the gauge group that algebra carries, $U(1) \times SU(2) \times SU(3)$. I state exactly where the derivation stops: at the entries of the finite Dirac operator, the operator on the internal space whose entries are the fermion masses and mixings.

The argument begins with the Root Tautology: interaction, comparison and relation are one concept, and that identification cannot be denied without being used. Six concepts are intrinsic to every comparison, in three inseparable pairs: sameness and difference, content and context, and the subjective and the objective. Sameness and difference, applied to content and context, give four compound concepts: continuity, symmetry, discontinuity and asymmetry. The inseparability of sameness and difference leaves exactly two ways of joining the compound concepts in one comparison: continuity with asymmetry, and symmetry with discontinuity. The

Incommensuration Theorem, a structural fact about the nature of knowing, states what the two surviving joinings entail: symmetry and continuity cannot both be perfected in one comparison, and asymmetry and discontinuity cannot both be perfected in one comparison. The three axioms are derived from the theorem.

From the theorem and the axioms, by steps each stated as a scope, a theorem or an identification, the book arrives at the comparison algebra. The GNS construction carries the comparison algebra to its Hilbert space, and the first four postulates are derived together with the Hilbert space. Gleason's theorem, with each of its premises traced to the structure of comparison, gives the Born rule. Superposition, entanglement, spin, interference, Bell nonlocality and the statistics of identical particles each follow from a named feature of the comparison algebra. The measurement problem dissolves, because the derivation removes the conditions under which the problem was formulated.

II. How the Steps Are Marked

Chapter 2 states the criterion against which the rest of the book is to be read. A derivation begins from premises that presuppose no part of the formalism it produces; each of its steps follows from the steps before it; it reaches the specific algebra of the actual world, or it states exactly where it stops; and it states every identification as an identification.

An identification is a stated step from the structure of comparison to a physical quantity or a mathematical condition, made together with the observation that bears on it. I name each theorem as a theorem and each identification as an identification, so that the reader can examine every step as the kind of step it is.

III. How the Book Is Arranged

Part I, *The Question*, sets out the six postulates and the question of their origin (Chapter 1), and the criterion a derivation must meet (Chapter 2).

Part II, *The Structure of Comparison*, presents the Root Tautology, the six intrinsics and the three modalities (Chapter 3), the Incommensuration Theorem and the three axioms (Chapter 4), and the steps from the axioms to the comparison algebra (Chapter 5).

Part III, *The Derivation of Quantum Mechanics*, constructs the Hilbert space (Chapter 6), derives the Born rule (Chapter 7), derives superposition, entanglement, spin and the other central quantum phenomena (Chapter 8), and dissolves the measurement problem (Chapter 9).

Part IV, *The Wider Landscape*, derives the qubit, the quantum gates and the no-cloning theorem (Chapter 10), and asks of each existing reconstruction and interpretive program which features of quantum mechanics that program assumes and which it derives (Chapter 11).

Part V, *The Passage*, traces each postulate to its source and maps the whole chain from comparison to quantum mechanics (Chapter 12).

Appendix A sets out in full the two steps that carry the most weight for a reader who checks the chain: the Three-Carrier Theorem, which fixes the form of the algebra, and the algebraic form of the Incommensuration Theorem. Appendix B states what the derivation says experiment will find.

Each chapter builds on the chapters before it, and the book is written to be read in order. Chapter 3 states the conceptual foundation, and Chapters 4, 5 and 6 carry that foundation to the comparison algebra and to its Hilbert space.

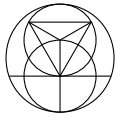
IV. The Apparatus of the Book

Each chapter opens with an abstract and closes with a summary, a box of its key concepts, and a list of the works it cites. Locations within the book are given by chapter and section: the third section of Chapter 5 is written 5.III, and the first section of Appendix A is written A.I.

The Terms section at the back gathers the definitions the book uses, each with the place where the reader first meets the term in that sense. The Bibliography gathers every work the chapters cite into one alphabetical list. The Index locates each term and name by chapter and section, and every location in the Index is a link to its section.

PART I

The Question



Chapter 1

The Axioms You Did Not Choose

On the Postulates of Quantum Mechanics and the Question of Their Origin

ABSTRACT

Every practicing physicist learns the postulates of quantum mechanics in the same way a musician learns the rules of harmony: as a set of prescriptions that produce correct results without revealing why those particular prescriptions, and no others, should govern the world. This chapter isolates the six standard postulates of non-relativistic quantum mechanics, states what each one asserts, and asks the question that textbooks leave unasked: *where did these axioms come from?* The postulates were not derived from anything more fundamental. They were assembled over about twenty-five years of experimental surprises and mathematical improvisation, and they contain within themselves a structural contradiction, the measurement problem, that no interpretation has successfully removed. The persistence of that contradiction is a symptom of a deeper issue: the postulates have no common source. The question that the remainder of the book answers is whether a common source can be found from which all six postulates follow as necessary consequences, together with the specific algebra and gauge group on which they are realized.

I. The Six Postulates

The standard formulation of non-relativistic quantum mechanics rests on six postulates. The numbering and precise wording vary from textbook to textbook, but the content is essentially the same whether one follows von Neumann, Dirac, Sakurai, or Weinberg. What follows is a composite statement that captures the structural content common to all standard treatments.

Postulate 1: The State Space

The state of a physical system is represented by a unit vector in a complex, separable Hilbert space H. The Hilbert space is a complete inner-product space over the complex numbers, and the inner product $\langle\psi|\varphi\rangle$ is a complex number satisfying the standard sesquilinear conditions.

This postulate is the most foundational of the six postulates and the one most often taken for granted. It asserts three things at once: that physical states live in a *vector space* (so that superposition is possible), that the scalars are the *complex numbers* (not the reals, not the quaternions), and that the space is *complete* in the topological sense (so that limiting procedures converge). None of these three features is explained by the postulate itself.

Postulate 2: Observables

Every measurable physical quantity corresponds to a self-adjoint (Hermitian) operator on the Hilbert space. The possible results of measuring the quantity are the eigenvalues of the operator.

Self-adjointness guarantees real eigenvalues, which is physically necessary if measurements are to yield real numbers. But the postulate does more than require reality: it asserts a specific algebraic structure for the observables. Bounded self-adjoint operators on a Hilbert space form a Jordan algebra (an algebra whose product is commutative but need not be

associative) under the symmetrized product $\frac{1}{2}(AB + BA)$, and generate a C*-algebra (an algebra of operators closed under adjoints and under limits in the operator norm) under multiplication and adjunction. This algebraic structure is rich, constraining, and entirely unexplained by the postulate.

Postulate 3: Time Evolution

The time evolution of a closed system is governed by the Schrödinger equation:

$$i\hbar \, d|\psi\rangle/dt = H|\psi\rangle$$

where H is the Hamiltonian operator. Equivalently, time evolution is unitary: the state at time t is obtained from the state at time t_0 by applying a unitary operator $U(t, t_0)$.

Unitarity preserves inner products, and therefore preserves probabilities. It is deterministic: given the state at one time, the state at all future times is completely determined. This determinism will prove important when we reach Postulate 5.

Postulate 4: Composite Systems

The state space of a composite system is the tensor product of the state spaces of its components: $H_{AB} = H_A \otimes H_B$.

Postulate 4 is the postulate that makes entanglement possible. The tensor-product space contains states that cannot be written as products of states from the component spaces. These non-separable states are the entangled states, and their existence is a direct consequence of the tensor-product rule.

The postulate is sometimes omitted from shorter presentations, but it is logically independent of the others. One could imagine a theory with Hilbert spaces, self-adjoint operators, and unitary evolution but with a different composition rule: a direct sum, for instance, rather than a tensor product. The choice of tensor product is a substantive physical assertion.

Postulate 5: Measurement Outcomes

When a measurement of observable A is performed on a system in state $|\psi\rangle$, the result is one of the eigenvalues a_k of A, and the state of the system immediately after the measurement is the corresponding eigenstate $|a_k\rangle$ (or, in the degenerate case, the projection of $|\psi\rangle$ onto the eigenspace of a_k, renormalized).

Postulate 5 is the notorious "collapse postulate". It asserts that measurement produces a discontinuous, non-unitary change of state. The system was in a superposition; after measurement, it is in a definite eigenstate. The transition is instantaneous, irreversible in practice, and governed by no dynamical equation.

Postulate 6: The Born Rule

The probability of obtaining eigenvalue a_k when measuring observable A on a system in state $|\psi\rangle$ is given by the Born rule:

$$p(a_k) = |\langle a_k|\psi\rangle|^2$$

The Born rule is the bridge between the mathematical formalism and experimental prediction. Without it, quantum mechanics would be a beautiful but physically vacuous piece of mathematics. The Born rule is what connects abstract Hilbert-space vectors to the frequencies observed in the laboratory.

The rule is quadratic in the amplitudes: it uses the modulus squared of the inner product, not the inner product itself, not the fourth power, not the logarithm. The specific functional form $|\langle\cdot|\cdot\rangle|^2$ is not derived from anything within the formalism. It is stipulated.

I take the postulates in their standard non-relativistic form, as given by von Neumann, Dirac, Sakurai, Weinberg and Landsman. The further structure of quantum field theory is reached later in the chain, where the algebra of Chapter 5 meets the spectral triple of Chapter 6 (an algebra, the

Hilbert space it acts on, and a Dirac operator that measures distance between states). The relativistic equation of the electron, the Dirac equation, and the antiparticles it requires are reached there too (Chapter 6, Section VII).

II. What the Postulates Do Not Explain

The six postulates are astonishingly successful. Together they predict the energy levels of hydrogen, the specific heat of solids, the statistics of photon bunching, the stability of matter, the existence and properties of antimatter, and thousands of other phenomena with a precision unmatched by any other physical theory. No experiment has ever contradicted them.

And yet there are questions that the postulates not only fail to answer but fail even to address. These questions are about the postulates themselves, and not about what happens at the Planck scale or in the vicinity of black holes: questions that arise the moment one asks not "do these rules work?" but "why *these* rules?"

Why a Hilbert Space?

A Hilbert space is a very specific mathematical structure. It is a vector space (so superposition is possible), it is over the complex numbers (not the reals, not the quaternions, not any other field or division algebra), it has an inner product (so angles and lengths are defined), and it is complete (so that Cauchy sequences converge).

Each of these features does real physical work. Superposition produces interference. Complex numbers give every amplitude a phase on a circle, not only a sign. The inner product supplies the amplitudes from which the Born rule computes probabilities. Completeness ensures the spectral theorem works.

But *why* this combination? Why not a real Hilbert space? Real quantum mechanics is a consistent theory; Stueckelberg formulated it in 1960. For a single system it reproduces every prediction of the complex theory, interference included. The two theories differ only in how independent systems compose. Why does nature choose the complex numbers over the reals?

The question is now experimental as well as formal. Renou and colleagues (2021) showed that real-number quantum theory and complex quantum theory make different predictions for a network of independent sources, and network experiments reported in 2022 found the predictions of complex quantum theory. Nature uses the complex numbers; the postulates still do not say why.

Similarly, why not a quaternionic Hilbert space? Quaternionic quantum mechanics has been investigated since the work of Finkelstein, Jauch, Schiminovich and Speiser (1959 to 1963), and in Adler's 1995 monograph. It is mathematically consistent but produces different physical predictions: predictions that, so far as experiment can tell, are not realized in nature.

The postulates do not explain why the complex numbers are selected. They simply assert it.

Why the Born Rule?

The Born rule $p(a_k) = |\langle a_k|\psi\rangle|^2$ is the only place in the formalism where probabilities appear. It is also the only place where the formalism makes contact with experiment. Everything before it is deterministic algebra; everything after it is frequentist statistics. The Born rule is the hinge.

Why the modulus squared? Why not the modulus itself, or the modulus to the fourth power, or some other function of the amplitude? The question is not idle. The modulus squared is not the only probability assignment that is consistent with the Hilbert-space structure. It is, however, the only one that depends only on the subspace and is additive on orthogonal subspaces: a fact established by Gleason's theorem in 1957.

Gleason's theorem shows that, in a separable Hilbert space of dimension three or more, any assignment of probabilities to subspaces that depends only on the subspace and is additive on orthogonal subspaces (countably additive, in infinite dimension) has the form $p(P) = \mathrm{tr}(\rho P)$ for a density operator ρ; for a pure state this is the Born rule. Gleason's theorem is a powerful result, but it pushes the question back one step: why should the probability of a subspace depend only on the subspace, and why should probabilities be additive on orthogonal subspaces? What physical principle demands the two conditions? The postulates are silent.

Why Linear Superposition?

The superposition principle (the assertion that if $|\psi\rangle$ and $|\varphi\rangle$ are possible states of a system, then any complex linear combination $\alpha|\psi\rangle + \beta|\varphi\rangle$ is also a possible state) is one of the most distinctive features of quantum mechanics. It is directly responsible for interference, for the double-slit pattern, for quantum computation, and for the very concept of quantum coherence.

But superposition is not a law of nature in the way that conservation of energy is a law. It is an assumption about the mathematical structure of the state space. It follows from the choice of a vector space as the arena for states. A different choice (a convex set, a lattice, a manifold without linear structure) would yield a different theory.

The postulates assert superposition by asserting Hilbert space. They do not explain why the state space should have linear structure.

Why the Tensor Product for Composites?

When two systems are combined, the state space of the composite is the tensor product of the component spaces. The tensor product is a very specific mathematical operation. It produces a space whose dimension is the *product* of the component dimensions, not the sum. And it permits entangled states: states of the composite that are not decomposable into states of the parts.

The alternative, a direct sum, would produce a state space whose dimension is the sum of the component dimensions, and it would not permit entanglement. Such a theory would describe systems that are always separable, that never share correlations beyond what classical probability allows. Experiment finds systems that share correlations beyond what classical probability allows.

But the postulates offer no reason for the tensor product over the direct sum, beyond the empirical fact that the tensor product gives the right answers. Entanglement, the most distinctive feature of quantum mechanics, is built in by fiat.

III. The Measurement Problem as a Symptom

The six postulates contain a structural contradiction. Postulate 3 says that time evolution is unitary: deterministic, linear, continuous, reversible. Postulates 5 and 6 say that measurement produces a discontinuous, non-unitary, irreversible change of state.

Both cannot be fundamental. If unitary evolution is the complete law of motion, then measurement is a physical process governed by that law, and no collapse should occur. If collapse is a fundamental process, then unitary evolution does not hold universally, and Postulate 3 is false during measurements.

The conflict between unitary evolution and collapse, which cannot both be fundamental, is the measurement problem. It has been discussed, debated, and agonized over for nearly a century, and it remains unresolved. Every interpretation of quantum mechanics (Copenhagen, Many-Worlds, Bohmian mechanics, QBism, consistent (or decoherent) histories, objective collapse, relational quantum mechanics) is an attempt to resolve this contradiction.

The Interpretations

The Copenhagen interpretation dissolves the problem by fiat: classical measuring devices are taken as given, and the formalism applies only to the system being measured. The Many-Worlds interpretation keeps Postulate 3 as absolute and reinterprets Postulates 5 and 6 as descriptions of branching: all outcomes occur, and the observer finds herself in one branch. Bohmian mechanics adds hidden variables (definite particle positions) and replaces Postulate 5 with a deterministic guidance equation. QBism reinterprets the quantum state as a personal degree of belief rather than a physical property, dissolving the problem by relocating it. Objective-collapse theories (GRW, Penrose) modify Postulate 3: unitary evolution holds approximately, but a collapse mechanism, stochastic or induced by gravity, intervenes at macroscopic scales.

Each interpretation resolves the contradiction at a price. Copenhagen pays with vagueness about the classical/quantum boundary. Many-Worlds pays with an extravagant ontology of unobservable branches. Bohmian mechanics pays with manifest non-locality and the loss of Lorentz invariance at the fundamental level. QBism pays by surrendering the idea that quantum mechanics describes an objective physical world. Objective collapse pays with new physics that has not been observed.

What is rarely noted is that the *existence* of the measurement problem is itself diagnostic. Postulates that are derived from a common source do not contradict one another. If Postulate 3 and Postulates 5/6 were consequences of a single deeper principle, the apparent contradiction between them would be resolved not by interpretation but by construction: the derivation itself would specify the domain of applicability of each, and the boundary between unitary evolution and measurement would emerge from the structure rather than being drawn by hand.

The fact that the postulates conflict is evidence that they were not derived from a common source. And indeed they were not. Postulate 1 comes from Hilbert's theory of integral equations (1906). Postulate 2

draws on von Neumann's spectral theorem and his axiomatization of 1932. Postulate 3 is Schrödinger's equation (1926), itself inspired by Hamilton's classical mechanics. Postulate 4 is the composition rule made explicit by von Neumann (1932) and Schrödinger (1935). Postulate 5 is von Neumann's projection postulate (1932), building on Heisenberg (1927) and Dirac (1930), with Lüders' rule (1951) for the degenerate case. Postulate 6 is Born's probabilistic interpretation of the wave function (1926).

The postulates were assembled from six different mathematical and physical traditions over about twenty-five years, from Hilbert's spaces of 1906 to von Neumann's axiomatization of 1932, with the later refinements of 1935 and 1951 named above. They were unified by success: they give the right answers. But they were never derived from a common starting point. The measurement problem is the scar left by that assembly.

IV. The Question This Book Answers

I pose a single question, in two parts. The first part of the question asks:

Can all six postulates of quantum mechanics be derived from something more fundamental?

A derivation of the general framework alone would still leave open why the quantum theory of the world is built on one particular algebra, with the gauge group of the Standard Model, $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, and not on another. So the question has a second part:

Can the specific algebra of the actual world, and the gauge group it carries, be derived from the same source?

If the answer is yes (if there exists a set of axioms from which Hilbert space, self-adjoint operators, unitary evolution, tensor products, the Born rule, and the measurement update all follow as consequences) then the

questions of the preceding section are answered automatically. Why complex numbers? Because the derivation produces them. Why the Born rule? Because the derivation entails it. Why the measurement problem? Because, from the deeper perspective, there is no problem: what looked like a contradiction was an artifact of starting from the wrong place.

The claim of this book is that such a derivation exists. The axioms from which it proceeds are not physical axioms in the ordinary sense. They are not statements about particles, fields, or forces. They are statements about the structure of comparison itself: about what is necessarily involved in any act of relating one thing to another. The framework from which the derivation proceeds is the Immanent Metaphysics (IM), a formal account of the structure of comparison, introduced in Chapter 3. The derivation shows that the specific mathematical structures of quantum mechanics (the Hilbert space, the algebra of observables, the Born rule, the tensor-product composition law) are not arbitrary choices or lucky guesses. They are the unique structures consistent with the requirements of comparison, through identifications that the book names at each step.

The chapters that follow will present the ground, construct the derivation, and trace it to the six postulates. This first chapter has served to make the question precise: we know what we are trying to derive, and we know what a successful derivation would look like. It would be a chain of reasoning that begins from a premise that cannot be denied without being used, proceeds by theorems and by identifications each named as such, and ends with the six postulates of quantum mechanics, together with the specific algebra and its gauge group.

V. Summary

The postulates of quantum mechanics were not chosen by anyone. They were assembled from six traditions over about twenty-five years, and they were unified by success, not by derivation. Their one internal contradiction, between unitary evolution and the update on measurement, is the symptom of that assembly.

The resulting theory is spectacularly successful but structurally opaque. Its postulates are axioms in the mathematical sense (starting points that are not derived from anything prior) but they are not axioms in the philosophical sense of self-evident truths. No one, encountering a Hilbert space for the first time, would say: "of course; it could not have been otherwise."

The question, then, is whether it *could* have been otherwise. Whether there is some deeper structural reason that the state space must be a complex Hilbert space, that the observables must be self-adjoint operators, that the probabilities must follow the Born rule. If such a reason exists, it would transform quantum mechanics from a collection of empirically successful rules into a consequence of something that can be understood on its own terms.

The remainder of this book is devoted to showing that such a reason exists.

KEY CONCEPTS IN THIS CHAPTER

- **Hilbert space**: a complete inner-product space over the complex numbers. The arena in which quantum states live, and the content of Postulate 1.
- **Self-adjoint operator**: an operator equal to its own adjoint, guaranteeing real eigenvalues. The mathematical representation of an observable, and the content of Postulate 2.
- **Unitary evolution**: time evolution that preserves inner products. Deterministic, linear, continuous, and reversible. The content of Postulate 3.
- **Tensor product**: the composition rule for combining the state spaces of separate systems. Produces entanglement. The content of Postulate 4.
- **Measurement update**: the replacement of the state by the eigenstate (or normalized projection) belonging to the outcome found. The content of Postulate 5.
- **Born rule**: the prescription $p = |\langle\cdot|\cdot\rangle|^2$ that converts amplitudes into probabilities. The bridge between formalism and experiment. The content of Postulate 6.
- **Measurement problem**: the structural contradiction between unitary evolution (Postulate 3) and the collapse upon measurement (Postulates 5/6). A symptom of the fact that the postulates were not derived from a common source.
- **Gleason's theorem**: the mathematical result (1957) that, on a separable Hilbert space of dimension ≥ 3, every assignment of probabilities to subspaces that depends only on the subspace and is additive on orthogonal subspaces has the form $\mathrm{tr}(\rho P)$; for a pure state, the Born rule.

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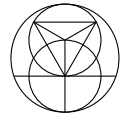
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Chapter 2

The Criterion of Derivation

What It Means to Derive a Physical Theory

ABSTRACT

Before a derivation can be read as one, the criterion of derivation must be established. This chapter distinguishes derivation from translation and reformulation, and states the criterion against which the rest of the book is to be read: premises that presuppose no part of the formalism they produce; steps that each follow from what precedes them; an endpoint that reaches the specific algebra of the actual world, not only the general framework of quantum theory, or states exactly where it stops; and every identification stated as one. Chapter 11 asks of each existing program which features of quantum mechanics it assumes and which it derives.

I. Derivation, Translation, Reformulation

The word *derivation* is used loosely in the physics literature. It is applied to translations, to reformulations, and to constructions that assume much of what they claim to derive. This imprecision matters, because if the criterion is unclear, the assessment of any claim to have derived something becomes a matter of taste rather than of logic. Three distinct activities must be separated.

A *derivation* begins with axioms and arrives at the target formalism with no unstated assumptions. Every step follows from what precedes it. A step that needs a premise not among the axioms and not derived from them must state that premise openly; a step from the structure of comparison to a physical quantity or condition is stated as an identification. A premise used silently is a smuggled assumption, and a chain that contains one is not a derivation.

A *translation* recasts an existing formalism in a new mathematical language. Phase-space mechanics translates Newtonian mechanics into Hamiltonian form; the path-integral formulation translates quantum mechanics into a sum over histories. Translations can be illuminating (they reveal structure that was invisible in the original language) but they are not derivations. The target formalism is present in the starting point; it has only been re-expressed.

A *reformulation* rearranges the postulates of an existing theory. Instead of starting from the six postulates of Chapter 1, one might start from a density-matrix formulation, or from an algebraic formulation in terms of C*-algebras, or from an information-theoretic formulation in terms of distinguishable states and channels. These reformulations are valuable (they clarify which postulates are independent and which are redundant) but they do not derive the theory. They assume it and reorganize it.

II. The Criterion Stated

The criterion of derivation can now be stated in full. A derivation of quantum mechanics must satisfy four conditions, the conditions that the book *Measurement; The Steps from Comparison to Physics* states for any derivation.

First, the axioms must not presuppose any part of the quantum formalism. They must be intelligible without reference to Hilbert spaces, observables, unitary evolution, tensor products, the Born rule, or measurement theory. If the axioms already contain these concepts in disguised form, the derivation is circular.

Second, every step of the derivation must follow from what precedes it. If a step requires a premise that is not among the axioms and has not been previously derived, that premise must be stated as a premise. Silent assumptions are not permitted.

Third, the derivation must arrive at the full structure of quantum mechanics: not only the general framework (Hilbert spaces, Born rule, unitarity) but the specific algebra on which that framework is realized, and the gauge group that algebra carries. A derivation that stops at the general framework does not answer the question of Chapter 1: why *this* quantum theory and not another. The particle spectrum that the algebra hosts is the subject of the wider program; I carry the derivation to the algebra and to the quantum formalism built on the algebra, and I state exactly where the derivation stops: at the entries of the finite Dirac operator, the operator on the internal space whose entries are the fermion masses and mixings.

Fourth, every step that is an identification must be stated as one. An identification is a stated step from the structure of comparison to a physical quantity or a mathematical condition, made with the observation that bears on it. Its reasons are given, and the reader can examine them. Identifications are a kind of step, not a grade of claim: a derivation names its theorems as theorems and its identifications as identifications, so that a reader who rejects one can see exactly which step is lost and what the remaining steps still prove.

Measurement puts the four conditions in one sentence: "A derivation begins from premises that presuppose no part of the target formalism; it proceeds through steps each of which follows from what precedes it; it arrives at the full structure of the target, or states exactly where it stops; and it states every step that is an identification as one."

The chapters that follow present the derivation. It begins with the Root Tautology (Chapter 3), establishes the Incommensuration Theorem and the three axioms (Chapter 4), arrives at the comparison algebra (Chapter 5), constructs the Hilbert space (Chapter 6), derives the Born rule (Chapter 7), derives quantum phenomena (Chapter 8), dissolves the measurement problem (Chapter 9), derives the qubit, the quantum gates and the no-cloning theorem (Chapter 10), and sets the result beside the existing reconstruction and interpretive programs and the route of Chamseddine and Connes, asking of each what it assumes and what it derives (Chapter 11). Chapter 12 traces the passage as a whole: where each postulate comes from, with a map of the whole chain, and what the passage changes. Appendix A shows two steps of Chapters 4 and 5 in full, and Appendix B states what the derivation says experiment will find.

III. Summary

The question posed in Chapter 1, *where do the quantum postulates come from?*, has been sharpened. A derivation answers it only if it begins from axioms that do not contain the answer, proceeds by steps that each follow, ends at the specific structure of the actual world or says where it stops, and names each identification as one. Translation and reformulation, however illuminating, do not answer it. The criterion of this chapter is the measuring rod for everything that follows.

KEY CONCEPTS IN THIS CHAPTER

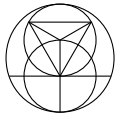
- **Derivation**: A chain of reasoning from axioms to target formalism in which every step follows from what precedes it, with no smuggled assumptions.
- **Translation**: The recasting of an existing formalism in a new mathematical language, as phase-space mechanics recasts Newtonian mechanics.
- **Reformulation**: A rearrangement of the postulates of an existing theory, which clarifies which postulates are independent without deriving the theory.
- **The criterion**: Four conditions: the axioms do not presuppose the target formalism; every step follows from its predecessors; the derivation reaches the specific algebra, or states where it stops; every identification is stated as one.
- **Identification**: A stated step from the structure of comparison to a physical quantity or mathematical condition, with the observation that bears on it; a kind of step, not a grade.

References

F. Landry, *Measurement; The Steps from Comparison to Physics* (UVSM Publications, 2026), Chapter 1.

PART II

The Structure of Comparison



Chapter 3

The Root Tautology and the Six Intrinsics

What Cannot Be Denied Without Being Instantiated

ABSTRACT

Chapter 1 asked whether the postulates of quantum mechanics can be derived from something more fundamental, and Chapter 2 set the criterion a derivation must meet. This chapter begins the answer by presenting the starting point of the Immanent Metaphysics (IM): the structure of comparison. The Root Tautology establishes that interaction, comparison, and relation are structurally identical concepts, and that this equation cannot be denied without self-contradiction. From this tautological fixed point, six concepts are shown to be intrinsic to comparison: three inseparable pairs (sameness/difference, content/context, subjective/objective) that are both necessary and sufficient for any act of relating one thing to another. The three modalities (omniscient, immanent, and transcendent) are irreducible roles that the intrinsic pairs instantiate. The three modalities are structural necessities of comparison itself, and not metaphysical categories imported from philosophy. The chapter is self-contained: it assumes no prior acquaintance with the Immanent Metaphysics and requires only a willingness to follow a conceptual argument with the same care one would give to a mathematical proof.

I. The Root Tautology

Consider three concepts that are ordinarily treated as distinct: *interaction*, *comparison*, and *relation*. The claim is that these three concepts are structurally identical, that they name the same act seen from three perspectives.

Every interaction is a comparison. To interact with something is necessarily to distinguish it from its background, to register a difference or a sameness between what is encountered and what was expected or remembered. A photon absorbed by an atom is a comparison between the photon's energy and the spacing of the atom's levels. A thermometer in a gas compares its temperature with the gas's: heat flows until the two agree, and the mercury records the result. Every physical interaction is an act of comparison in the precise sense that it distinguishes one configuration from another. The application to physics is an identification, the first of the book: every finding of physics is identified with an act of comparison made from inside the world.

Every comparison is a relation. To compare two things is to place them in correspondence, to establish a mapping between the properties of one and the properties of the other. The result of a comparison is a statement of relation: this is greater than that, this is the same as that, this differs from that in such-and-such a way.

Every relation is an interaction. A relation that connects no terms, that is registered by nothing, that has no instantiation in any process, is not a relation in any operative sense. It is a notation on paper, devoid of content. A relation becomes real, becomes a relation in the full sense, only when it is instantiated in some interaction between the things it relates.

This threefold equation is tautological in the precise logical sense. It cannot be denied without self-contradiction. Any attempt to deny that interaction is comparison requires *comparing* the two concepts, which is it-

self an *interaction*. The denial instantiates the very structure it attempts to negate. The act of arguing against the equation is an instance of the equation.

> **For the Physicist**
>
> The Root Tautology may seem like word-play, but the Root Tautology is not word-play. The Root Tautology is a *fixed point*: an assertion that maps to itself under its own application. The physicist will recognize the structure: a fixed point of a map is a point x such that $f(x) = x$. The Root Tautology says that "comparison" is a fixed point of the operation "compare this concept with that concept." When you compare "comparison" with "interaction" or "relation," you get comparison back. The return of comparison to itself is a structural observation about the self-referential closure of the concept of comparison.

The Root Tautology is the foundation of everything that follows. It is, deliberately, the most minimal possible starting point: it assumes less than any physical or mathematical premise, and it is substantive, since the six intrinsics, the four compound concepts, the Incommensuration Theorem and the three axioms are all entailed by it. No empirical observation can refute it, because any empirical observation is itself a comparison (between a measurement and a prediction, between a detector reading and a calibration). No logical argument can refute it, because any logical argument is itself a comparison (between premises and conclusions, between a claim and its negation). The tautology is the ground floor: the one thing that cannot be doubted without simultaneously performing it.

II. The Six Intrinsics

If comparison is a real act (and the Root Tautology establishes that it cannot coherently be denied) then we can ask: what is structurally involved in any act of comparison? What concepts must be present whenever anything is compared with anything?

The answer is six concepts, organized in three pairs.

Sameness and Difference

Every comparison produces a verdict: the things compared are alike in some respects and unlike in others. Without both sameness and difference, comparison cannot occur. If there were only sameness, there would be nothing to compare: the two things would be identical in every respect, and the act of comparison would have no content. If there were only difference, there would be no *basis* for comparison: the two things would share nothing, and no statement of relation could connect them.

This need for both sameness and difference is familiar to the physicist. A measurement that always gives the same result regardless of what is measured carries no information. A measurement whose outcomes share no common scale with any other measurement is meaningless. Information arises at the boundary between sameness and difference.

Content and Context

Every comparison has something it is about, and a respect in which the two things compared are two. Content is what a comparison is about, and context is the respect in which the two things compared are two.

A tide gauge in a harbor compares the height of the water at noon with its height at midnight (*Category Theory and the Immanent Metaphysics*, Chapter 1, Section II). The height of the water is the content; noon and midnight are the context. In a laboratory, the spin found is the content, and the two runs, two times or two settings being compared are the context. Without content, there is nothing to measure. Without context, there are not two things to compare, and a reading has no meaning: a number that stands against no standard and no second reading is not a measurement.

Content and context are inseparable. Where there is content, there must be context within which the content is situated. Where there is context, there must be content that the context frames. Neither can exist without the other.

Two premises govern content and context (*Comparison; The Necessary Basis*; restated in *Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 1, Section II). The law of symmetry: symmetry applies to all that is of content, so that what is found as content is found as a sameness of content across a difference of context. The law of continuity: continuity applies to all that is of context. Content is local and mutually exclusive, two contents excluding one another from the same place; context is non-local and composes with other context. Chapter 4 derives the conjugacy of Axiom I from the two laws, and Chapters 5, 7 and 8 use them again.

Subjective and Objective

Every comparison has a perspective from which it is performed and that which is compared. The terms "subjective" and "objective" are used here in a structural sense, not a psychological one. The subjective is the side that performs the comparison; the objective is the side that is compared.

In a laboratory experiment, the apparatus is the subjective and the system being studied is the objective. In a scattering experiment, the probe is the subjective and the target is the objective. The terminology does not imply consciousness or mind; it implies only the structural asymmetry between the comparator and the thing compared.

Like the other pairs, subjective and objective are inseparable. A comparator with nothing to compare is not performing a comparison. An object with no comparator registering its properties is, as Chapter 4 puts it in stating Axiom I, indistinguishable from nothing.

Necessity and Sufficiency

The claim is that these six concepts (sameness, difference, content, context, subjective, objective) are both necessary and sufficient for the formal consideration of comparison.

Necessary: removing any one prevents comparison from occurring. Without sameness, no basis for relation. Without difference, no information. Without content, nothing that the comparison is about. Without context, no respect in which the two things compared are two. Without a subjective perspective, no act of comparing. Without an objective target, nothing to compare.

Sufficient: no seventh concept is needed beyond what these six already provide. Any candidate for a seventh intrinsic can be shown to be a compound of two or more of the existing six; the formal argument is given in *Comparison; The Necessary Basis*, Chapter 4. Causality, for instance, is a specific pattern of sameness and difference across a specific content/context partition. Information is a measure of the distinction between sameness and difference. None of these requires a new fundamental concept beyond the six.

The Thought Experiment

The necessity and sufficiency of the six intrinsics can be grounded physically through a thought experiment. Imagine an otherwise empty universe containing only five particles: two hydrogen atoms (each consisting of a proton and an electron) and one photon. One atom emits the photon, and the other atom absorbs the photon. Read from the emitting atom, the emitting atom is the subjective and the absorbing atom the objective; read from the receiving atom, the two exchange roles, since the subjective and the objective are found by perspective (*Comparison; The Necessary Basis*, Chapter 4, Section VI). The particles are the content; the empty universe is the context. The before-and-after orbital states provide the sameness and the difference.

The exchange of the photon between the two atoms is the simplest conceivable complete comparison: five particles, six concepts, and nothing redundant. The thought experiment does not prove the sufficiency claim, only the formal argument of *Comparison; The Necessary Basis* does that, but it shows that even at the most minimal physical level, comparison requires all six concepts and no fewer.

III. The Three Pairs as Structural Properties

The three pairs share three structural properties. Understanding these properties is essential for the algebraic derivation that follows in later chapters.

Distinctness. The members of each pair have different meanings. Sameness is not difference. Content is not context. Subjective is not objective. Distinctness is obvious, but I state distinctness explicitly because the algebraic derivation of later chapters depends on distinctness.

Inseparability. Wherever one member of a pair appears, the other is necessarily implicit. There is no sameness without difference, no content without context, no subjective without objective. Inseparability is the

property that carries the most mathematical weight. Inseparability means that the six intrinsics are three constrained pairs, rather than six independent variables.

Non-interchangeability. The three pairs themselves cannot be substituted for one another without altering the structure of comparison. The sameness/difference pair is not the same kind of distinction as the content/context pair, which is not the same kind as the subjective/objective pair. The three pairs play different structural roles.

> **Mathematical Resonance**
>
> The physicist will recognize these three properties (distinct, inseparable, non-interchangeable) as the defining characteristics of a *non-trivial decomposition of a structure into irreducible components*. In the algebraic derivation of Chapter 5, the three properties, which Axiom III states of the three modalities, become a condition on the simple summands of an algebra: each summand carries one modality (distinct), every modality has a summand (inseparable), and no two summands have the same algebraic character (commutative, non-commutative division, or non-division), which is non-interchangeability. That step is an identification, stated as one, and with it the algebra has exactly three summands.

IV. The Three Modalities

The three pairs of intrinsics play three different roles with respect to one another, and the Immanent Metaphysics calls the roles modalities. Each role is defined by its position in the act of comparison. Named by position in a single comparison, the immanent is the act of comparing, the omniscient is the thing compared, and the transcendent is the thing compared against. The thing compared against and the stance from which a comparison is made are one role seen from two sides: the standard against which a comparison is made is the position from which it is made, as a ruler is both what a length is set against and where the measurer stands.

The Omniscient Modality

The omniscient is the role of what is laid out as structure: a pattern, a map, a record, what can be held in view as a whole and shared. In physics at large, structure and the conservation laws are omniscient.

A Hilbert space is an omniscient-modal object. A Lagrangian is an omniscient-modal object. A theorem is an omniscient-modal object. Anything that can be written down, stored in a database, or communicated in a textbook is an expression of the omniscient modality. What is laid out as structure exists: it is the same in every instance, like the pattern of a law or the state of a system at one instant.

The Immanent Modality

The immanent is the role of the act itself, as it happens, here and now. In physics at large, interaction and force are immanent.

A scattering event is immanent-modal. A measurement is immanent-modal. The moment of comparison itself (the instant at which the probe interacts with the target) is immanent-modal. What is given in an actual act of comparison is real: it occurs once, here and now, with the future still open.

The Transcendent Modality

The transcendent is the role of the stance and the possibility from which an act proceeds: the position from which a comparison is made, with the range of what is possible from it. In physics at large, possibility and the conditions within which events occur are transcendent.

A preparation procedure is transcendent-modal. A choice of basis is transcendent-modal. The decision to measure spin along the z-axis rather than the x-axis, and the range of outcomes that this setting makes possible, are expressions of the transcendent modality. The evaluative character often given to the transcendent is the character of that position: the stance is where a selection is made from among what is possible.

Which pair takes which role

Which pair of intrinsics stands in which role depends on the focus of the argument being made; the concepts alone do not fix the roles (*An Immanent Metaphysics*, aphorism 1.41-2). A focus is what a comparison is being considered as. In the focus of measurement, which takes comparison as it is performed by an instrument, sameness and difference are immanent, since the verdict is given in the act; content and context are omniscient, since they are the finding laid out as structure; and the subjective and the objective are transcendent, since they are the side from which the measurement is made and the side about which it is made. Of the six ways of assigning three pairs to three roles, exactly three keep the direction of the cycle of Axiom II, stated in the next chapter; they are rotations of one another. The focus of measurement takes one; the focus of comparison as relation, in which the subjective and the objective are immanent, takes another.

> **Why "Modalities" and Not "Categories"**
>
> The terminology may grate on the physicist's ear. "Omniscient" sounds theological; "transcendent" sounds mystical. The names designate roles, and they carry no theological meaning. They are intended to be transparent to the structural role each modality plays: the omniscient is what is laid out and can be known whole, the immanent is what is happening, the transcendent is the stance from which something can happen. Alternative terminologies have been proposed: structural/processual/creative; pattern/interaction/potential; being/becoming/selection. All name the same three roles. The IM uses the modality names because they have specific technical definitions within the system.

The Running Example: One Silver Atom

One experiment is followed through this book: a silver atom sent through the magnet of Gerlach and Stern. The magnet, turned to measure spin along one axis, is the stance, with its range of two possible outcomes (transcendent). The passage of the atom through the field, and its deflection, is the act (immanent). The spot on the screen is the record of the finding, laid out as structure (omniscient). In the focus of measurement, the apparatus is the subjective and the atom the objective; the spin found is the content, and the orientation of the magnet is the context; the verdict, up or down, is a sameness or a difference. Chapters 6, 7, 9 and 10 take the same atom further.

V. For the Physicist: What Comes Next

The material presented in this chapter is the conceptual foundation. It establishes the starting point: comparison as a tautological given, six intrinsics as the necessary and sufficient concepts involved in comparison, and three modalities as the irreducible roles that these concepts instantiate.

None of this, so far, is mathematics. It is conceptual analysis: a statement of the structural elements that must be present in any act of comparing anything with anything. The physicist may be impatient: where are the equations? where is the derivation?

The answer is that the derivation requires one more ingredient: the Incommensuration Theorem and the three axioms. I take up the Incommensuration Theorem and the three axioms in Chapter 4. Together with the six intrinsics, and through the identifications that Chapter 5 states, they will produce the specific algebraic structure, the comparison algebra $A_{\mathrm{F}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, from which the Hilbert space, the gauge group, the Born rule, and the complete formalism of quantum mechanics will be derived.

The modalities are not mystical categories. They correspond to specific mathematical structures. In the focus of the two laws, the immanent modality is carried by $\mathbb{H}$, the quaternion algebra; Chapter 5 proves this as a theorem from the conjugacy condition, which reads Axiom I in that focus. In the frame of measurement of *The Story of the Particles*, a rotation of the same labeling, $\mathbb{H}$ carries the transcendent. The frame of measurement is the set of units that *The Story of the Particles* uses, the frame "in which the comparing act is light" (*The Kinds of Particle*, Chapter 4, Section III). In the focus of the two laws, the omniscient modality is carried by $\mathbb{C}$, the complex numbers, and the transcendent by $M_3(\mathbb{C})$, the algebra of 3×3 complex matrices; Chapter 5 states this labeling as an identification and gives the four considerations that fix it.

What remains to be established is the derivation itself: the chain of reasoning that begins with the conceptual foundations of this chapter and arrives at the specific algebra through a sequence of constrained steps. That chain is the subject of Chapters 4, 5, and 6.

VI. Summary

The Root Tautology and the six intrinsics are the ground floor of the Immanent Metaphysics. They are not physical axioms. The Root Tautology is the premise of the whole ground, and the six intrinsics are what it entails; both are statements about what is structurally involved in the most basic act of which any physical system, any mind, any formal system is capable, the act of comparing one thing to another.

This starting point has a distinctive character. It assumes less than any physical premise (it does not presuppose spacetime, particles, fields, or forces), less than any mathematical premise (it does not presuppose numbers, sets, or topology), and yet it is not vacuous. Six specific concepts have been set out. Three modalities have been named. Structural properties have been established. These will prove to be enough (combined with the theorem and the axioms of the next chapter, and with the identifica-

tions that Chapter 5 names) to determine a unique finite-dimensional algebra, and from that algebra, the complete formalism of quantum mechanics.

The reader trained in physics may find it surprising that anything useful can follow from so minimal a starting point. The remainder of the book is devoted to showing that it does.

KEY CONCEPTS IN THIS CHAPTER

- **Root Tautology**: the equation of interaction, comparison, and relation as structurally identical concepts. A fixed point that cannot be denied without self-contradiction.
- **Intrinsics of comparison**: the six concepts (sameness, difference, content, context, subjective, objective) that are both necessary and sufficient for any act of comparison.
- **Inseparability**: the structural property that wherever one member of a pair (e.g. sameness) appears, the other (difference) is necessarily implicit.
- **Non-interchangeability**: the property that the three pairs play structurally different roles and cannot be substituted for one another.
- **Content and context**: what a comparison is about, and the respect in which the two things compared are two.
- **Law of symmetry**: symmetry applies to all that is of content; what is found as content is found as a sameness of content across a difference of context (*Comparison; The Necessary Basis*).
- **Law of continuity**: continuity applies to all that is of context (*Comparison; The Necessary Basis*).
- **Omniscient modality**: the role of what is laid out as structure; in a single comparison, the thing compared.
- **Immanent modality**: the role of the act itself, here and now; in a single comparison, the act of comparing.
- **Transcendent modality**: the role of the stance and the possibility from which an act proceeds; in a single comparison, the thing compared against.
- **Focus**: what a comparison is being considered as; which pair of intrinsics takes which role depends on it.

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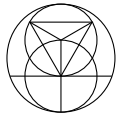
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Chapter 4

The Incommensuration Theorem and the Three Axioms

Why Symmetry and Continuity Cannot Both Be Perfected, and How the Three Modalities Are Related

ABSTRACT

The previous chapter set out the six intrinsics of comparison and the three modalities. This chapter states the central result of the Immanent Metaphysics, the Incommensuration Theorem, and the three axioms derived from it. Sameness and difference applied to content and context give four compound concepts: continuity, symmetry, discontinuity, and asymmetry. Of the ten conjunctions of two compounds, the inseparability of sameness and difference, applied in each respect, excludes eight. Two survive: continuity with asymmetry, and symmetry with discontinuity. Symmetry and continuity cannot both be perfected in one comparison, and neither can asymmetry and discontinuity. The two surviving modes are identified with the two characters of physics: continuously asymmetric spacetime, and a finite internal space whose findings are discontinuously symmetric, composed side by side in the product $M \times F$. A physical theory carries the two characters at its two levels, its findings and its formalism. The axioms are quoted in their exact wording and derived from the theorem in four steps.

I. The Four Compound Concepts

From the six intrinsics of comparison, four compound concepts can be constructed. With the stance of the comparing (the subjective and the objective) held the same, they are formed by applying the sameness/difference pair to the content/context pair. The definitions of the four compound concepts are precise:

Definitions

Continuity: a sameness of content where there is a sameness of context. Nothing is sudden.

Symmetry: a sameness of content where there is a difference of context. Something is invariant.

Discontinuity: a difference of content where there is a sameness of context. Something is sudden.

Asymmetry: a difference of content where there is a difference of context. Nothing is invariant.

These definitions are not metaphorical. They are constructed entirely from the intrinsics of comparison. Each compound is specified by a pair of values (sameness or difference) assigned to each of two roles (content and context). If I abbreviate sameness as ‘s’ and difference as ‘d’, each compound can be expressed as a pair:

Compound	Content	Context	Code
Continuity	sameness	sameness	(s, s)
Symmetry	sameness	difference	(s, d)
Discontinuity	difference	sameness	(d, s)
Asymmetry	difference	difference	(d, d)

The physicist will recognize these concepts immediately. Symmetry in physics means that the laws give the same results in different reference frames: sameness of content, difference of context. Continuity in physics means that a smooth change in initial conditions produces a smooth change in outcomes: sameness of content, sameness of context. The IM definitions are not alternative definitions of these concepts; they are the structural specifications of what these concepts *mean* when grounded in the intrinsics of comparison.

Read at the finest grain, continuity is a proportionality between the change of context and the change of content, not a constancy. Physics also calls groups, functions and manifolds symmetric or continuous; that is a second usage, of mathematical objects, and the theorem concerns the first, the character of what a comparison finds.

II. The Incommensuration Theorem

The Question

The four compound concepts are not identical to one another. There exist discrete, non-continuous things that are symmetric, and twisted, continuous things that are not symmetric. So we can ask which conjunctions of two compound concepts can characterize one comparison, held exactly at its foundation. Can something be both symmetric *and* continuous? Both asymmetric *and* discontinuous?

Four compounds pair in ten ways: each with itself, which gives four pairings, and each with each of the others, which gives six. A conjunction of two compounds carries two verdicts on the content and two verdicts on the context. The enumeration of the ten candidates and the three criteria is given in *An Immanent Metaphysics*, Technical Appendix, aphorisms 1R1 to 1R20.

The Three Criteria

Criterion 1 (*no self-application*). A compound concept applied to itself adds nothing to what the compound already says. The criterion removes continuity with continuity, symmetry with symmetry, discontinuity with discontinuity, and asymmetry with asymmetry.

Criterion 2 (*no opposite pairing*). Continuity with discontinuity gives opposite verdicts on the content (same, different) under one and the same verdict on the context (same, same). Symmetry with asymmetry gives opposite verdicts on the content under one and the same verdict on the context (different, different). In each, one context receives two contradictory content verdicts, which violates the distinctness of sameness and difference, and the criterion removes both. In the two survivors, each content verdict is paired with a different context verdict.

Criterion 3 (*the balance of sameness and difference*). Sameness and difference are inseparable wherever a verdict is given. A conjunction of two compounds gives its verdicts in two respects, content and context, so the inseparability holds within each respect: among the verdicts on the content, sameness must appear with difference, and among the verdicts on the context, sameness must appear with difference. The criterion uses inseparability exactly as the six intrinsics state it, and it is read in each respect separately.

The Ten Candidates

Conjunction	Verdicts on content	Verdicts on context	Verdict
Continuity with asymmetry	same, different	same, different	**survives**
Symmetry with discontinuity	same, different	different, same	**survives**
	same, same		

Conjunction	Verdicts on content	Verdicts on context	Verdict
Continuity with symmetry		same, different	forbidden: no difference of content (Criterion 3)
Discontinuity with asymmetry	different, different	same, different	forbidden: no sameness of content (Criterion 3)
Continuity with discontinuity	same, different	same, same	removed (Criterion 2; also fails Criterion 3)
Symmetry with asymmetry	same, different	different, different	removed (Criterion 2; also fails Criterion 3)
Each compound with itself (four)	one verdict twice	one verdict twice	removed (Criterion 1; also fails Criterion 3 in both respects)

The balance criterion alone removes continuity with symmetry and discontinuity with asymmetry, the two pairings the theorem forbids; neither of the other two criteria removes them. The balance criterion also fails the six candidates that the other two criteria remove, so inseparability alone, applied in each respect, reaches the same two survivors. The other two criteria are kept because they name the reasons the self-pairings and the two opposite pairings fail.

The same ten candidates can be laid out as a grid. Each compound heads one row and one column; each cell above the diagonal is one conjunction, and each cell on the diagonal is a compound with itself. The cells below the diagonal repeat the cells above it and are left empty.

	Continuity	Symmetry	Discontinuity	Asymmetry
Continuity	itself	forbidden	opposite	**survives (CA)**
Symmetry		itself	**survives (SD)**	opposite
Discontinuity			itself	forbidden
Asymmetry				itself

"Itself" marks the four self-pairings (Criterion 1), "opposite" the two opposite pairings (Criterion 2), and "forbidden" the two pairings that only the balance criterion removes (Criterion 3). The two survivors are the two cells whose compounds differ both in their verdict on the content and in their verdict on the context.

The Incommensuration Theorem

"The concepts of symmetry and continuity cannot both be simultaneously and fundamentally applied to any event or comparison. The concepts of asymmetry and discontinuity cannot both be simultaneously and fundamentally applied to any event or comparison. Any comparison, perfected at its foundation, is either continuously asymmetric or discontinuously symmetric."

The theorem is a structural fact about the nature of knowing. It is proved in *Incommensuration; The Theorem and the Three Axioms*, Chapter 6; the wording above is that book's premise P9, which writes "eventity", an event and an entity under one name (*Incommensuration*, Chapter 6, Section VII), where the wording above writes "event". The closing sentence is as *Measurement; The Steps from Comparison to Physics*, Chapter 1, states it. The theorem has no physical premise, so it applies to every domain in which comparison occurs. "Perfected" means holding exactly, at the foundation of a description. Symmetry and continuity appear together approximately, as they do in every real system; what the theorem forbids is the two holding exactly in what one and the same comparison finds, and the same of asymmetry and discontinuity.

The Logic of the Step

The proof is a finite case analysis over ten candidates, and every case is examined. The three criteria are argued from the inseparability of sameness and difference, which is a statement about what comparison is. The same rule can also be run as a tally across the whole conjunction (*Incommensuration*, Chapter 6, Section VI). On the two forbidden pairings the tally gives the verdict that the reading in each respect gives: continuity with symmetry carries three samenesses against one difference, and discontinuity with asymmetry three differences against one sameness. The tally is not the whole test: symmetry with symmetry also carries two of each and fails the balance in both respects, so a tally read by itself needs the first criterion beside it, and the reading in each respect needs nothing beside it. The theorem has an exact finite form, in which the admissible comparisons are the two symmetries of the set {same, different}, the identity and the exchange (*Category Theory and the Immanent Metaphysics*, Chapter 5), and an exact algebraic form, in which an observable that is both central and commutes with the Dirac operator can report only which class of comparison a state lies in (*Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 5). Here an observable is central when every inner symmetry of the algebra leaves it unchanged; the Dirac operator is the operator that measures difference between states; and a class of comparison is a set of pure states joined to one another by steps of finite distance or of inner symmetry. Chapter 5, Section II states the first two exactly.

III. The Two Modes as Physics

The Incommensuration Theorem yields two modes: the continuously asymmetric (CA) and the discontinuously symmetric (SD: symmetric, discontinuous; not the code (s, d) of the compound symmetry). I make three identifications, each a stated step from the structure of comparison to physics: the continuously asymmetric mode with the character of

spacetime; the discontinuously symmetric mode with the character of what is found of the internal space; and, in the box at the end of the section, the conjugation of Axiom I with the Fourier relation between position and momentum.

The CA Mode: External Spacetime

The CA mode is continuous (smoothly varying, without jumps) and asymmetric (every point is distinguishable from every other). The CA mode has the structure of a smooth manifold, the mathematical object that models spacetime in general relativity. Its points form one connected whole, and a generic metric has no symmetry at all: "A generic solution of Einstein's equations has no Killing vector field" (*Measurement*, Chapter 9, Section III).

Gravity is identified with the asymmetric structure of the continuously varying spacetime manifold. The curvature of spacetime is the expression of asymmetry within the continuous domain. The CA mode is the structural specification for what physicists call the external, gravitational sector of physics.

The SD Mode: The Internal Space

The SD mode is symmetric (exactly invariant) and discontinuous (with sharp boundaries between the regions of invariance). In physics it is the character of the internal space, the finite geometry on which the gauge groups U(1), SU(2) and SU(3) act. That geometry falls into separate classes of comparison, two in the Standard Model: the electroweak class, made of the one pure state of $\mathbb{C}$ and the one pure state of $\mathbb{H}$, and the color class, a copy of $\mathbb{C}P^2$. Its symmetries act on it exactly. What is found of it is discrete and exact: charges take definite values, and color is found only in color-neutral combinations.

The observables of the internal space are not all of one character. Each observable of the finite algebra falls into one of four cells, according to whether it is symmetric (central) and whether it is continuous (commutes

with the Dirac operator). Any two color states are carried into one another by a symmetry, and together they form a continuum; the observables that tell them apart lie in the continuously asymmetric cell. What is found of color is discontinuously symmetric, since only color-neutral states are found (*Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 5, Section IV).

Two Levels of One Theory

Every physical theory has two levels: its formalism, which exists in the exact sense of Chapter 3 (laid out as structure, the same in every instance), and its findings, which are real in the exact sense of Chapter 3 (given once, in an actual act of comparison). A law exists; a measurement is real. The theorem concerns what a comparison finds, so it applies at the level of the findings, and a theory can carry the two characters at its two levels (*Category Theory and the Immanent Metaphysics*, Chapter 1, Section VIII). *Measurement* states it for content: "The findings of content, at their foundation, are discontinuously symmetric, and the formalism that describes them rests on continuous structures" (*Measurement*, Chapter 9, Proposition 9.3(a)).

Quantum mechanics is the clearest case. Its findings are discontinuously symmetric: discrete outcomes, exact charges, definite eigenvalues. Its formalism rests on continuous structures: a Hilbert space whose states combine with continuous complex coefficients. Because quantum mechanics is discontinuously symmetric in the findings and continuous in the formalism, superposition and discrete outcomes appear in one theory without contradiction. They belong to its two levels.

The Product Geometry

Both modes, the continuously asymmetric and the discontinuously symmetric, are present in physics, and the two modes are composed side by side, not merged:

$$M \times F$$

where M is a continuous asymmetric manifold (external spacetime) and F is a finite internal space whose findings are discontinuously symmetric. In Connes's program of non-commutative geometry, M is a four-dimensional spin manifold, and F is a finite non-commutative space described by a finite-dimensional algebra. Connes uses the product geometry for the Standard Model coupled to gravity. The conjunction is mediated by scope, as the foundations allow: "symmetry and continuity may both appear in one description when they are separated by level, by phase, or by scope" (*Incommensuration*, Chapter 6). The product geometry is a conjunction of the two characters, mediated by scope, and not a single geometry perfecting both characters, which the theorem forbids.

The Uncertainty Principle as an Instance of the ICT

The Heisenberg uncertainty principle states that position and momentum cannot both be known with arbitrary precision. Position and momentum are conjugate descriptions of one state, related by a Fourier transform. That transform is identified with the conjugation of Axiom I, stated below: each description can be perfected only at the expense of the other. The momentum description carries symmetry: momentum is the generator of translations, and a state of definite momentum is unchanged by a translation, up to a phase. The position description carries continuity: position is a coordinate of the continuum, and a state of definite position marks one point of it. A state sharply localized in position is spread in momentum, and the reverse; a description that perfected both at once would conjoin symmetry with continuity, which the theorem excludes. The uncertainty principle is an instance of the Incommensuration Theorem. The bound $\hbar/2$ follows, by the Robertson inequality, from the canonical commutator $[x, p] = i\hbar$ on $L^2(M, S)$, which Chapter 6, Section V derives.

IV. The Three Axioms

The three pairs of intrinsics play three different roles, the modalities of Chapter 3: the omniscient, the role of what is laid out as structure; the immanent, the role of the act itself; and the transcendent, the role of the stance and the possibility from which an act proceeds. Three axioms relate the three roles. They are given here in the exact wording of *An Immanent Metaphysics*, aphorism 1.3-1.

The Three Axioms

Axiom I. "The immanent is more fundamental than the omniscient and/or the transcendent. The omniscient and the transcendent are in conjugate relationship to one another."

Axiom II. "A class of the transcendent will precede an instance of the immanent. A class of the immanent will precede an instance of the omniscient. A class of the omniscient will precede an instance of the transcendent."

Axiom III. "The (classes/instances) of the immanent, omniscient, and transcendent are distinct, inseparable, and non-interchangeable."

Five terms have an exact sense. A class is a kind, standing ready as a whole. An instance is one particular case of a kind, here and now. In Axiom I, "and/or" means of either one and of both together: the immanent is prior to each of the other two and to both at once. In Axiom III, the bracket means that the statement holds both of the classes and of the instances. Two things are conjugate when each can be perfected only at the expense of the other, and neither is prior to the other in definitional basis.

In plain words, Axiom I says that the act comes first, and that structure and stance are conjugate. Axiom II says that the stance comes first, then the act, then the record of the finding, laid out as structure, and that this laid-out record shapes the next stance. Axiom III says that the three roles are distinct, that they are inseparable, and that none can do the work of another. The order of Axiom II is one of precedence and not of clock time: the cycle T → I → O → T (transcendent, immanent, omniscient) can be entered at any point, and its direction cannot be reversed.

V. From the Theorem to the Axioms

The axioms are consequences of the theorem, and *Incommensuration*, Chapter 8, derives each of them from it. The derivation has four steps.

First, the conjugacy of Axiom I is the theorem said of the roles. In the focus of the two laws (Chapter 3, Section II; *Incommensuration*, Chapter 1, Section III), the law of symmetry applies to all that is of content, the law of continuity applies to all that is of context, and symmetry carries the omniscient character and continuity the transcendent. By the theorem the two cannot both be perfected, so the omniscient and the transcendent are conjugate. Each conjugate takes the one mode the theorem admits for it: the omniscient is discontinuously symmetric, the transcendent continuously asymmetric.

Second, the act comes first because both conjugates are made from it. Neither the omniscient nor the transcendent is prior to the other, and each is defined from the act of comparison in which they stand. That the act comes before both conjugates, and that neither conjugate comes before the other, is the priority clause of Axiom I.

Third, the conjugacy fixes the direction of Axiom II. A process that involves both conjugates must pass between them in sequence, through the act, and each forward step of the cycle is many to one: a class of many possibilities leads to one instance. The reverse step would be one to many, which is a relation and not a passage. The same order is found in the way the pairs of intrinsics engage in a comparison: each step takes a pair, a class, to one member of the next pair, an instance; in the focus of measurement the order runs from the subjective and the objective (the stance) to sameness and difference (the verdict given in the act) to content and context (the finding laid out).

Fourth, the three pairs of intrinsics over which the theorem is stated give Axiom III. The theorem is stated over all three pairs: it applies sameness and difference to content and context, and it holds the subjective and the ob-

jective fixed as the stance. These three pairs are distinct, inseparable, and none can stand in for another; the roles they play inherit the same three properties.

What Axiom I says in physics

In physical terms, interaction is more fundamental than existence. Every process of establishing that something exists depends on interaction: measurement, observation, detection. An entity with no physical properties, no detectable presence by any instrument, no influence on any field or any spectrum, is indistinguishable from nothing; and what no comparison can distinguish is not two things but one. Existence without any interaction is therefore non-existence, and the act of interaction is prior to the existence it establishes.

> **The Lightning Bolt**
>
> The image of the lightning bolt illustrates Axiom I. Watching a thunderstorm, one treats the lightning bolt as more fundamental than both the ground and the sky. The image has physical content. The solidity of matter (the fact that hands cannot pass through each other) rests on electromagnetism and the exclusion principle: the bond between nucleus and electron is an electric process, and what is experienced as solid existence is built from it. Through lightning, the ground senses the sky, and the sky senses the ground. Their very being is a relationship.

What Axiom II says in physics

The simplest physical comparison shows the cycle: two hydrogen atoms and the photon that passes between them. This frame is a different focus from the focus of measurement of Chapter 3, Section IV, and in it the roles fall as follows. The emitting atom, with its possible transitions, is the stance from which the exchange proceeds (transcendent). The emission, passage and absorption of the photon is the interaction itself (immanent). The changed state of the receiving atom is the record of the finding, laid

out as structure (omniscient), and that structure is the condition of the next exchange. The cycle runs T → I → O → T, and each forward step is many to one: many possible transitions lead to one emission, and many emissions to the same final state.

What Axiom III says in physics

No modality can be removed without collapsing the domain. A domain without structure would have nothing to interact with; a domain without process would be a frozen ledger, structure without dynamics; a domain without stance and possibility would have no initial conditions and nothing from which an event could proceed. And no two can exchange roles: structure and stance play different parts in the cycle, and exchanging them reverses the order of precedence.

The axioms in finite form

Category Theory and the Immanent Metaphysics, Chapter 5, states the axioms exactly as properties of an oriented triangle of three terms. Axiom III: there are three terms (distinct), they form a single cycle (inseparable), and the only symmetries admitted are the three rotations of the cycle, so no exchange of two terms is admitted (non-interchangeable). Axiom II: the cycle has an orientation, T → I → O → T. Axiom I: the omniscient and the transcendent are exchanged by one reflection of the triangle, and the immanent is the one term that reflection leaves in place. That reflection reverses the orientation. So the omniscient and the transcendent are conjugate, and still not interchangeable, because the only map exchanging them runs Axiom II backwards. The immanent's priority is its position as the fixed point of their conjugacy.

VI. Self-Description

The axioms satisfy their own conditions. They are three statements that are distinct, inseparable, since none can be removed without destroying the framework, and non-interchangeable, since exchanging them changes the meaning. Axiom I, a structural overview of how the modalities are arranged, has the character of the omniscient. Axiom II, which describes the directed movement of the cycle and enacts it, has the character of the immanent. Axiom III, which sets boundary conditions that hold from beyond any single frame, has the character of the transcendent.

Each axiom is also one edge of the cycle. *An Immanent Metaphysics*, aphorism 1.3-3, reads each axiom as a consideration in one modality of the nature of another: Axiom III, an omniscient consideration of the nature of the transcendent; Axiom II, a transcendent consideration of the nature of the immanent; Axiom I, an immanent consideration of the nature of the omniscient. These three pairs are exactly the three edges O → T, T → I and I → O.

The axioms applied to themselves give a second description of the framework, and applying them once more gives nothing new: the axioms close at the second order of self-description, and the third order adds nothing (*Incommensuration*, Chapter 16). That closure is the answer of the Immanent Metaphysics to the regress of justification: the framework does not justify itself by repeating its claims, but by instantiating its own structure in the act of being examined.

> **For the Physicist**
>
> Self-descriptive closure may seem unusual to readers trained in physics, where axioms are external prescriptions imposed on a theory. The structure has an analogue in mathematical logic, where a system can describe its own syntax. Here the closure is not a circularity: it is the way the system meets the regress of its own justification.

VII. Summary

The Immanent Metaphysics rests on one theorem and three axioms derived from it. The Incommensuration Theorem shows that of the ten conjunctions of two compound concepts only two survive: continuity with asymmetry, and symmetry with discontinuity. Symmetry with continuity, and asymmetry with discontinuity, cannot both be perfected in one comparison. The axioms state that the act is prior and that structure and stance are conjugate (Axiom I), that stance, act and laid-out finding follow one another in a directed cycle (Axiom II), and that the three roles are distinct, inseparable and non-interchangeable (Axiom III).

The two surviving modes are identified with the two characters of physics: the continuous asymmetric manifold of spacetime, and the finite internal space, whose findings are discontinuously symmetric. Physics composes them side by side in the product $M \times F$. A physical theory carries the two characters at its two levels, and quantum mechanics is the clearest case: its findings are discrete and exact, and its formalism is a continuous space of superposed states.

What remains is to derive the algebra of the internal space, $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, from the structure of comparison, through the theorem, the axioms and the identifications that the next chapter states. I derive the algebra of the internal space in Chapter 5.

KEY CONCEPTS IN THIS CHAPTER

- **Compound concepts**: continuity (s, s), symmetry (s, d), discontinuity (d, s) and asymmetry (d, d), each a pair of verdicts on content and on context.
- **Balance criterion**: in each respect, content and context, sameness must appear with difference; alone it leaves the same two survivors.
- **Incommensuration Theorem (ICT)**: symmetry and continuity cannot both be perfected in one comparison, nor asymmetry and discontinuity; a comparison perfected at its foundation is continuously asymmetric or discontinuously symmetric.
- **CA mode**: the continuously asymmetric mode, identified with the character of spacetime.
- **SD mode**: the discontinuously symmetric mode (symmetric, discontinuous), identified with the character of what is found of the internal space.
- **Two levels**: a theory's formalism exists and its findings are real, in the exact senses of Chapter 3; the theorem applies to the findings.
- **M × F**: the two modes composed side by side, the conjunction mediated by scope.
- **Axiom I**: the immanent is prior to the omniscient and the transcendent, which are conjugate.
- **Axiom II**: the directed cycle T → I → O → T, each step many to one.
- **Axiom III**: the three modalities are distinct, inseparable and non-interchangeable.
- **Self-description**: the axioms have the characters of the omniscient, immanent and transcendent, and each is one edge of the cycle.

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Chapter 5

From Axioms to the Comparison Algebra

The Algebraic Route to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

ABSTRACT

This chapter carries the axioms of the Immanent Metaphysics to the finite-dimensional algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The chain has seven steps: comparison encoded as a *-algebra; the internal part of the product geometry as the scope; semisimplicity, from the faithful action of a complex context of evaluation (an identification); the Artin-Wedderburn and Frobenius theorems; the Three-Carrier Theorem, in which Axiom III read on the summands and the conjugacy clause of Axiom I read as complex type give $\mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C})$ and force the quaternions as the immanent carrier; the labeling of $\mathbb{C}$ as omniscient and $M_n(\mathbb{C})$ as transcendent; and re-projection, which fixes $n = 3$, confirmed by the closure $c_1^2 + N^2 = H^2$ with the triangle $\{3, 4, 5\}$. Each step is stated as what it is: a scope, a theorem, or an identification, each identification with the observation that bears on it. The steps contain no continuous parameter.

I. Why an Algebra?

The question that opens this chapter is the question that opens the entire derivation: why should the structure of comparison be encoded as an algebra? The answer is not that algebras are convenient, or that physics

happens to use them, or that the mathematician's toolbox contains them. The answer is that comparison itself has the structure of a *-algebra, and the six intrinsics give each part of that structure.

Consider the most elementary requirements. Comparisons compose: if one can compare *A* to *B*, and *B* to *C*, then one can compare *A* to *C*. This is the product operation. The composition is associative: in a chain of comparisons *A* to *B*, *B* to *C*, and *C* to *D*, the result is the same whether one first composes *A*-to-*B* with *B*-to-*C*, or first composes *B*-to-*C* with *C*-to-*D*.

Composition is partial: *A*-to-*B* composes with *B*-to-*C* and not with *C*-to-*D*. An algebra carries a partial composition in a standard way. A groupoid is a collection of objects with reversible arrows between some pairs of them, and in the algebra of a groupoid the product of two comparisons is their composite when the middle terms agree, and zero when they do not; weighted sums of comparisons are its elements, the degrees of comparison being the weights. Quantum mechanics began with the algebra of a groupoid. Heisenberg's rule for multiplying arrays of transition amplitudes is the product of the algebra of the groupoid of transitions between energy levels (A. Connes, *Noncommutative Geometry*, 1994, Chapter I, Section 1).

Comparisons reverse. The six intrinsics include both *subjective* and *objective*: the perspective and the target. Reversing which is which (exchanging the one who compares with what is compared) is an involution on the space of comparisons. In algebraic terms, this is the *-operation: a map that sends each element to its adjoint, satisfying (ab)* = b*a* and (a*)* = a. The reversal of a composed comparison reverses the order: to undo comparing *A*-to-*B*-then-*B*-to-*C*, one first reverses the *B*-to-*C* comparison, then reverses the *A*-to-*B*.

Comparisons can be scaled and summed. The *sameness/difference* pair from the six intrinsics permits comparisons to register degree: two items can be more alike or less alike. This introduces a linear structure: one can form weighted combinations of comparisons.

Taken together, these three structural features (associative composition, an antilinear involution reversing order, and a linear structure compatible with both) define a *-algebra. The encoding is the first identification of the chain: a transcription of what comparison already is into a language in which it can be made precise. The reversal is a genuine feature exactly when the algebra is non-commutative: if the involution also preserved the order of every product, every two elements would commute, and exchanging the one who compares with what is compared would change nothing.

> **Method note: algebraic encoding and the GNS.**
> The claim that comparison has *-algebraic structure is made precise by the GNS construction (Gel'fand-Naimark-Segal), which guarantees that any C*-algebra can be faithfully represented as bounded operators on a Hilbert space; Section III shows that the algebra of this chapter is one. The GNS construction will be the subject of Chapter 6; for the present chapter, the *-algebraic structure is the starting point. What matters here is the derivation of the specific algebra, not the passage from algebra to Hilbert space.

II. The Internal Part and Its Scope

The Incommensuration Theorem, stated in Chapter 4, leaves two modes of perfected structure: the continuously asymmetric (the CA mode) and the discontinuously symmetric (the SD mode). Physics composes them side by side in the product $M \times F$ of a spacetime manifold M and a finite internal space F.

The manifold factor carries the arena. Its algebra is the commutative algebra $C^\infty(M)$ of smooth functions on spacetime, and its character is carried by its states and its symmetries, not by its observables, which all commute. The points of M form one connected whole, joined at finite distance

with no gap: that is the continuity of the context. A generic metric has no isometry, so no symmetry of the algebra preserves the geometry: that is its asymmetry.

The finite factor carries the internal degrees of freedom. I work on the finite factor alone: its algebra A_F is finite-dimensional because it is the algebra of the internal part F of the product. The restriction to the finite factor is the scope of this chapter, and I state the restriction as a scope, not as a theorem. What is found of the internal part is discontinuously symmetric: its geometry falls into separate classes of comparison, and its symmetries act on it exactly.

Within the finite algebra, the four compound concepts have an exact operator reading, proved in *Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 5. Here an observable is a self-adjoint element of the real algebra; Chapter 6 extends the word to the self-adjoint operators of the complex algebra that A_F generates on its module, which for $\mathbb{H}$ include the Pauli matrices. An observable is symmetric when it is unchanged by every inner symmetry of the algebra, that is, when it lies in the center; it is continuous when it commutes with the Dirac operator, the self-adjoint operator on the Hilbert space of the internal part that measures difference: its commutators with observables measure how far apart two states lie, and its entries are the masses and mixings of the fermions. The algebraic Incommensuration Theorem then says that an observable both symmetric and continuous is constant on every class of comparison: it can report which class a state lies in, and nothing else. An observable of the CA mode must commute with the Dirac operator without being central, and such observables exist only in a non-commutative algebra. So the algebra of the internal part must be non-commutative.

III. Semisimplicity

Every comparison is evaluated in a context. Algebraically, the context of evaluation is a representation: a Hilbert space on whose vectors the observables are evaluated. The context of evaluation is a complex Hilbert space on which the observables act faithfully. The faithful complex context of evaluation is an identification. Faithfulness reads the requirement that comparison detect every difference it makes: distinct observables must act differently, since a representation that identified two of them would erase a difference that comparison must detect. Complex scalars read the conjugacy clause of Axiom I: a conjugate pair carried by the context needs a conjugation on it, a map that is antilinear and not linear, and a space has one only if its scalars are not real. Among scalars that are not real, the quaternions are set aside by composition. The premise used here is the identification of Chapter 6, Section VI, which is stated before any tensor product exists: two systems are two comparisons performed together without disturbing each other, which together exhaust the joint comparison. Over the complex numbers that premise is met, and the tensor product follows from it as a theorem; over the quaternions it is not met, since quaternionic Hilbert spaces have no tensor product of the standard kind for independent systems (S. L. Adler, 1995). So the premise of composition selects the complex numbers, and the theorem of Chapter 6 then builds the product on them. The observation that bears on it: real and complex quantum theory differ in how independent systems compose, and experiment finds the complex composition (M.-O. Renou and colleagues, 2021; Chapter 8, Section V).

A theorem of operator algebra settles the rest. A finite-dimensional *-algebra with a faithful *-representation on a finite-dimensional complex Hilbert space is a C*-algebra, and it is semisimple. The proof is short. The obstruction to semisimplicity is a nilpotent ideal closed under the involution; if r lies in it, r*r is a self-adjoint nilpotent operator, which must be

zero, and then the norm of r is zero, because $\|r\|^2 = \|r^*r\|$. A nilpotent comparison, one that ceases to register any distinction under repetition, cannot be carried by a faithful context of evaluation.

The algebras of this chapter are real algebras represented on complex Hilbert spaces, because the quaternions are a real algebra and not a complex one.

IV. The Artin–Wedderburn Decomposition

The Artin-Wedderburn theorem states that every finite-dimensional semisimple real *-algebra decomposes, uniquely up to reordering, as a direct sum of simple summands:

$$A \cong M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \dots \oplus M_{n_k}(D_k)$$

where each D_i is a real division algebra and each $M_n(D)$ is the algebra of $n \times n$ matrices with entries in D. The summands are determined by the algebra itself.

The theorem of Frobenius (1878) fixes which division algebras can appear: the only associative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, of real dimensions 1, 2 and 4; $\mathbb{R}$ and $\mathbb{C}$ are commutative, and $\mathbb{H}$ is not. Both theorems are proved, and neither owes anything to the Immanent Metaphysics.

Associativity is what excludes the octonions. The octonions are a real division algebra of dimension eight, and the exceptional Jordan algebra of three-by-three octonionic matrices is the one simple formally real Jordan algebra whose product comes from no associative algebra (Jordan, von Neumann and Wigner, 1934; Albert, 1934). Both fail the associativity of composed comparison (Section I), so neither can carry comparison, however closely their symmetries resemble those of the Standard Model. The proposals that build the Standard Model on the octonions or on the excep-

tional Jordan algebra (Dixon, 1994; Furey, 2016; Dubois-Violette and Todorov, 2019; Boyle, 2020) therefore begin from a structure that comparison does not have.

At this stage, A_F is a direct sum of simple matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The number of summands, the kind of each, and the size of each are still open, and the axioms fix them.

> **The quaternions are unique.**
> The Frobenius theorem shows that the quaternions are the only associative real division algebra that is not commutative. When the theorem of the next section requires a non-commutative division algebra, the choice is already made: it can only be $\mathbb{H}$.

V. The Three-Carrier Theorem

Three characters

Simple summands differ in the kind of comparison their observables support. A simple summand $M_n(D)$ has exactly one of three characters:

commutative, if $n = 1$ and D is $\mathbb{R}$ or $\mathbb{C}$;

non-commutative division, if $n = 1$ and $D = \mathbb{H}$;

non-division, if $n \geq 2$.

$M_n(D)$ is a division algebra exactly when $n = 1$, since for $n \geq 2$ the matrix units are divisors of zero; by Frobenius the division algebras are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, and only $\mathbb{H}$ is not commutative. So every simple summand has one character and only one, and the second character has exactly one member.

The characters have a reading in terms of comparison. In a commutative summand every observable is symmetric: it registers an invariant, classifying distinction, with no internal articulation. In the non-commutative division summand elements fail to commute, yet every non-zero element

is invertible: every change can be undone within the summand, which is the character of process. In a non-division summand there are proper projections, observables that select a part and discard the rest, and a whole projective space of pure states ($\mathbb{C}P^{n-1}$ when $D = \mathbb{C}$), the range from which each finding selects one: the character of the stance and the possibility from which an act proceeds.

Two identifications

Two clauses of the axioms are given the form of conditions on the summands. Each is an identification, and each carries the observation that bears on it. Call the simple summands the *carriers*. The conjugacy clause of Axiom I is read twice in this chapter: in Section III on the context of evaluation, where it makes the scalars complex, and here on the carriers, where it makes the omniscient and transcendent carriers of complex type. One clause does both pieces of work.

The carrier condition, which reads Axiom III. Each carrier carries exactly one modality, each modality has at least one carrier, and no two carriers have the same character. The first clause reads the distinctness and inseparability of the modalities: each modality is present in the algebra, and nothing in the algebra is outside the modal structure. The second reads non-interchangeability: two summands of one character could stand in for one another in every role that character supports, and the condition forbids it. It concerns roles, not sizes: $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ has two summands of one character, though no symmetry of the algebra exchanges them.

The conjugacy condition, which reads the conjugacy clause of Axiom I. The carriers of the omniscient and of the transcendent are of complex type. A summand is of complex type when its irreducible module is not equivalent to its complex conjugate, of real type when the two are equivalent through a conjugate-linear map squaring to $+1$, and of quaternionic type when they are equivalent through one squaring to -1. The condition reads the conjugate relationship of the omniscient and the transcendent as carried by their carriers: each comes with a conjugate distinct from it. The

immanent, which Axiom I places prior to the conjugate pair rather than within it, is not constrained. The observation that bears on it: in the finite geometry of the Standard Model, $\mathbb{C}$ acts on up-type states through λ and on down-type states through its conjugate $\bar{\lambda}$, two inequivalent modules, and $M_3(\mathbb{C})$ acts on color.

Theorem 5.1 (the Three-Carrier Theorem)

Let A_F be a finite-dimensional real *-algebra, faithfully represented on a finite-dimensional complex Hilbert space, satisfying the carrier condition and the conjugacy condition. Then

$$A_F \cong \mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C}), \quad n \geq 2,$$

the immanent carrier is $\mathbb{H}$, and the omniscient and transcendent carriers are $\mathbb{C}$ and $M_n(\mathbb{C})$, in an order the theorem does not fix.

The proof is a few lines. By the second clause of the carrier condition and the three characters, A_F has at most three carriers, one of each character. By the first clause there are three modalities, each with a carrier, so A_F has exactly three carriers: a commutative summand, $\mathbb{R}$ or $\mathbb{C}$; the summand $\mathbb{H}$; and a summand $M_n(D)$ with $n \geq 2$. $\mathbb{H}$ is of quaternionic type, so by the conjugacy condition $\mathbb{H}$ carries neither the omniscient nor the transcendent, and therefore $\mathbb{H}$ carries the immanent. The other two carriers are of complex type: the commutative one is $\mathbb{C}$, since $\mathbb{R}$ is of real type, and the non-division one is $M_n(\mathbb{C})$, since $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ are of real and quaternionic type. The full proof, with an enumeration of every triple of small summands, is in *Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 7.

Two sizes are fixed by the characters themselves. The quaternionic rank is not chosen: a matrix algebra over $\mathbb{H}$ of size two or more has the non-division character, so the only quaternionic summand of division character is $\mathbb{H}$ itself. The commutative summand is a field, of size one.

And the field is not chosen: $\mathbb{R}$ is excluded by the conjugacy condition, so the commutative carrier is $\mathbb{C}$, whose unit group is the circle of phases. Only the size n of the non-division carrier remains open, and Section VII fixes it.

The real dimension N of the immanent carrier is the first of the three integers of the modal structure, c_1, N and H, to be fixed:

$$N = \dim_{\mathbb{R}} \mathbb{H} = 4.$$

VI. The Labeling of the Carriers

In the focus of the two laws, the theorem fixes $\mathbb{H}$ as the immanent carrier. It leaves open which of $\mathbb{C}$ and $M_n(\mathbb{C})$ carries the omniscient and which the transcendent. The labeling is an identification, made in the focus of the two laws, in which "symmetry carries the omniscient character and continuity the transcendent" (*Incommensuration*, Chapter 1, Section III). Four considerations point the same way.

The cells. In the finite geometry of the Standard Model, every non-zero observable of the $\mathbb{C}$ carrier is central and fails to commute with the Dirac operator, so the $\mathbb{C}$ carrier lies wholly in the discontinuously symmetric cell. Every element of the $M_3(\mathbb{C})$ carrier commutes with the Dirac operator, so that carrier lies wholly on the continuous side, continuously asymmetric except for its scalars.

The modal operations. The three modalities can also be read as three operations on a represented algebra: the omniscient takes A to its center, what no inner change can alter; the immanent is A itself; the transcendent takes A to the algebra $B(\mathcal{H})$ of all operators on the Hilbert space in which A is represented, everything the context of evaluation can hold. $\mathbb{C}$ is the only carrier contained entirely in the center. $M_n(\mathbb{C})$ is the only carrier that is the full operator algebra of an irreducible module of dimension two or more, as $B(\mathcal{H})$ is for the transcendent operation.

The characters. A commutative carrier registers only invariant, classifying distinctions, which is the character of the omniscient. A non-division carrier has a continuum $\mathbb{C}P^{n-1}$ of pure states, the range of the possible, which is the character of the transcendent.

The roles. The three roles are those "of what is laid out as structure, of the act itself, and of the stance and the possibility from which an act proceeds". $\mathbb{C}$ has one pure state, only real scalars as observables, and no inner symmetries: an invariant label and nothing else, laid out as structure. $\mathbb{H}$ has only real scalars as observables, but its unit quaternions act, by inner symmetries, on a sphere of rays that no self-adjoint element of the real algebra $\mathbb{H}$ distinguishes: it shows itself only in what it does, as the act does. $M_3(\mathbb{C})$ has observables, inner action, and a whole space $\mathbb{C}P^2$ of pure states, the range from which a finding selects one.

The labeling is therefore:

$$\text{Omniscient} \to \mathbb{C}, \quad \text{Immanent} \to \mathbb{H}, \quad \text{Transcendent} \to M_n(\mathbb{C}).$$

No theorem rests on the labeling: the integers of the next section are defined by the algebraic type of a carrier, and their names follow the labeling. The labeling is not free, however. A labeling that rotates every label one step along the cycle of Axiom II keeps the orientation of the cycle, in a different framing. A labeling that exchanges two labels reverses it: the exchange of the omniscient and the transcendent carries $T \to I \to O \to T$ to $O \to I \to T \to O$ (*Category Theory and the Immanent Metaphysics*, Proposition 5.10). The omniscient and the transcendent are conjugate, and still not interchangeable: the only map that exchanges them runs Axiom II backwards (Chapter 4, Section V).

VII. The Rank Three

The integers of the modal structure

Two integers measure the carriers, each in the way that carrier is naturally measured. The transcendent integer $c_1 = n$ is the rank of $M_n(\mathbb{C})$, the number of independent directions in which the content of its module can vary. The immanent integer $N = 4$ is the number of real directions of the quaternions, the dimension of the process.

Re-projection

At the microscopic limit of the domain, the finest grain at which anything is found, the form of what can be found is the form of finding itself. The three positions of comparison (the three modalities, the roles in which every comparison stands) are repeated inside the content as its labels. I state the repetition of the three positions inside the content as an identification on the algebra:

> **Re-projection**
> The internal degrees of freedom of the content repeat the structure of comparison. The complex rank n of the module of the non-division carrier $M_n(\mathbb{C})$ (the transcendent carrier) equals the number of modalities, so $n = 3$.

Re-projection says that color is the three positions of comparison inside the content. The observation that bears on it: there are three colors. The same identification, read for the instances of the content rather than for its labels, bears on the number of generations; *The Kinds of Particle*, Chapter 7, Sections I and III, carries that reading.

Closure

The same integer is reached a second way. The transcendent and immanent contributions are independent components of one closed whole, the omniscient's account of the modal structure, the whole pattern taken at once, and independent components compose in quadrature, as the Hilbert-Schmidt norms of operators on orthogonal sectors add in squares. The closure is exact when the whole, the structural depth H, is an integer:

$$c_1^2 + N^2 = H^2, \quad H \text{ a positive integer.}$$

With $N = 4$ the closure fixes c_1 at once. It reads $(H - c_1)(H + c_1) = 16$, with both factors positive and of the same parity. The factorizations of 16 into two even factors are $2 \cdot 8$ and $4 \cdot 4$. The first gives $H = 5$ and $c_1 = 3$; the second gives $c_1 = 0$, which is no carrier. So the closure alone fixes $n = 3$, and under re-projection it holds as arithmetic: $9 + 16 = 25$.

Theorem 5.2 (the rank three)

Under the carrier condition, the conjugacy condition and re-projection, $A_F \cong \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, the structural constants are $(c_1, N, H) = (3, 4, 5)$. Under the closure in place of re-projection, the algebra and the constants are the same.

Two consequences follow. The immanent integer is one more than the transcendent, $N = c_1 + 1$; its reading in the particle content, the lepton as a fourth color, is carried by *The Connes Correspondence*, Chapter 17, Sections II and III. And the depth H exceeds the transcendent integer by $H - c_1 = 2$, the rank of the module of the quaternions.

Three routes to the triangle

Non-Commutative Geometry and the Immanent Metaphysics, Chapter 11, reaches {3, 4, 5} by three routes under the closure, each from different premises. The first route is the one I have given above, from $N = 4$. The second uses no algebra at all. The radius of the circle inscribed in an integer right triangle is its slack, $(c_1 + N - H)/2$, and the slack is always a positive integer; exactly one integer right triangle has slack 1, and every larger slack belongs to two triangles or more. So a closure fixes its constants without ambiguity exactly when the slack is 1, and then, with c_1 the shorter leg, the triangle is (3, 4, 5). The third applies the closure across the family of algebras that Chamseddine and Connes classify, through identifications that read the lepton as a fourth color; it belongs with that reading, and *The Connes Correspondence*, Chapter 17, Section III, carries it. The equation $c_1^2 + (c_1 + 1)^2 = (c_1 + 2)^2$ also has the single positive solution $c_1 = 3$; written without the last relation, as $c_1^2 + (c_1 + 1)^2 = H^2$, it has infinitely many, for example $20^2 + 21^2 = 29^2$, so the equation is a consequence of the closure and not its source.

Where the cycle lives

The cycle of Axiom II is not carried by the Dirac operator. It lives in the Hilbert space: the carriers act on it from the left and from the right, every two carriers share states on which both act (a condition the Standard Model's representation satisfies), and with three carriers the pattern of shared states is a triangle, a single cycle through all three (Chapter 6, Section III). The Dirac operator of the Standard Model acts along one edge of that triangle only, between the immanent and the omniscient carriers, the edge on which mass is generated, and it never touches color. The order-one condition on the Dirac operator (the condition that the Dirac operator change the carrier on at most one side of a state) requires the Dirac operator to act along that one edge only and never to touch color. The order-one condition is also why the Standard Model has no colored scalar fields. The orientation of the cycle comes to the algebra from the Incommensuration Theorem, as Chapter 4 derives it; the algebra carries the cycle and not its direction.

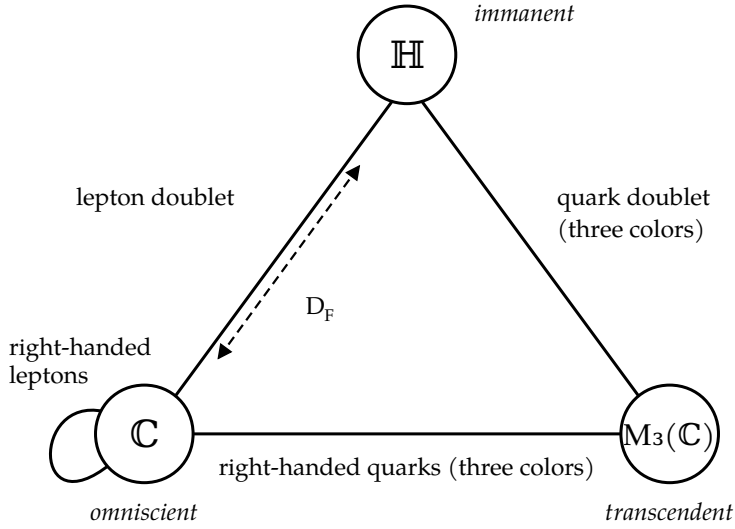


Figure 5.1. The three carriers and the sectors they share. Each sector of the Standard Model is named by the carrier acting on a state from the left and the carrier acting from the right: the lepton doublet joins $\mathbb{H}$ and $\mathbb{C}$; the quark doublet joins $\mathbb{H}$ and $M_3(\mathbb{C})$; the right-handed quarks join $\mathbb{C}$ and $M_3(\mathbb{C})$; and the right-handed leptons join $\mathbb{C}$ with itself. The finite Dirac operator D_F exchanges $\mathbb{H}$ and $\mathbb{C}$ on one side of a state, along one edge only, and never changes color.

VIII. The Comparison Algebra

Theorem 5.3 (the comparison algebra)

The algebra of the internal part of comparison is

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

with $\mathbb{C}$ the omniscient carrier, $\mathbb{H}$ the immanent carrier, and $M_3(\mathbb{C})$ the transcendent carrier. Its gauge group, the group of its unitaries reduced by the unimodularity condition, is $U(1) \times SU(2) \times SU(3)$, up to a finite group.

States and rays

As a real algebra, $\mathbb{C}$ has a single state, and so has $\mathbb{H}$, because the only self-adjoint quaternions are the real numbers. The pure states of $M_3(\mathbb{C})$ form the complex projective plane $\mathbb{C}P^2$. The richer geometry of the quaternions appears in their module, not in their states: $\mathbb{H}$ acts on $\mathbb{C}^2$, and the space of complex lines in $\mathbb{C}^2$, its ray space, is the sphere $\mathbb{C}P^1 = S^2$, on which the unit quaternions act by rotations. That sphere is the Bloch sphere (Chapter 8, Section IV).

The gauge group

A unitary of A_F is a triple (λ, q, m), with λ a phase, q a unit quaternion and m a unitary 3×3 matrix, so the unitary group of A_F is $U(1) \times SU(2) \times U(3)$. On the Hilbert space of the Standard Model, described in Chapter 6, a unitary acts from both sides of each state at once. The unimodularity condition of Chamseddine, Connes and Marcolli requires the determinant of that action, taken over the whole Hilbert space, to be one. It ties the phase of the $U(3)$ factor to the phase of the $\mathbb{C}$ factor, and leaves $U(1) \times SU(2) \times SU(3)$, up to a finite group: $SU(3)$ from $M_3(\mathbb{C})$, $SU(2)$ from $\mathbb{H}$, and $U(1)$ from $\mathbb{C}$, which has no inner symmetries of its own and enters only through its action on the two sides of a state. The hypercharges of the Standard Model follow, given its representation: they are computed from that representation and not put in.

The unimodularity condition is an input, and two readings are given for it in *Non-Commutative Geometry and the Immanent Metaphysics*. The first reading of the unimodularity condition is that the gauge transformations preserve the volume of the context of evaluation. The second reading of the unimodularity condition is that the comparing act is not one of the things that the comparing act compares, and carries no label. So the labels that the comparing act assigns sum to zero, and the hypercharges of each generation do sum to zero. Neither reading is a premise of a theorem.

The internal space

The ray space S^2 of the quaternions and the pure state space $\mathbb{C}P^2$ of the transcendent carrier together form the internal space $\mathbb{C}P^2 \times S^2$. The inner symmetries of A_F are exactly the rotations of that space: the group $SO(3) \times PU(3)$, of dimension 11, is the identity component of its group of isometries. The gauge group has one dimension more, the $U(1)$ of the omniscient carrier, which acts through the two sides of a state and moves no point of the space. So the gauge group is not the isometry group of $\mathbb{C}P^2 \times S^2$; it is that group together with the phase of the omniscient carrier.

IX. The Derivation Step by Step

The steps that carry comparison to the algebra are these, each stated as what it is. A scope says what the step covers; a theorem is proved from what precedes it; an identification is a stated step from the structure of comparison to a condition on the algebra, with the observation that bears on it. The kinds are kinds of step, and they are not grades of claim.

Step 1, the algebra. Comparisons compose, reverse and combine, so observables form a *-algebra. *Identification* (Section I).

Step 2, finite dimension. The algebra is that of the internal part of the product $M \times F$. *Scope* (Section II).

Step 3, semisimplicity. The context of evaluation is a complex Hilbert space on which the observables act faithfully. *Identification* (a faithful complex context of evaluation), then *theorem* (Section III).

Step 4, the summands. Artin-Wedderburn and Frobenius. *Theorems of mathematics* (Section IV).

Step 5, three carriers of three characters, with $\mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C})$ and $\mathbb{H}$ immanent. *Theorem*, from the carrier condition (Axiom III) and the conjugacy condition (Axiom I), which are *identifications* (Section V); Appendix A sets out every candidate the two conditions exclude.

Step 6, the labeling of $\mathbb{C}$ and $M_n(\mathbb{C})$. *Identification* in the focus of the two laws; no theorem rests on it (Section VI).

Step 7, $n = 3$. *Theorem*, from re-projection, an *identification*; confirmed by the closure, a second identification (Section VII).

The steps contain no continuous parameter. What they leave open is named: the entries of the finite Dirac operator, which carry the masses and mixings of the fermions. The derivation stops there. The full chain, with every proof and the observation that bears on each identification, is *Non-Commutative Geometry and the Immanent Metaphysics*, Chapters 4 to 12; its Theorem 12.2 shows that the Standard Model's own finite geometry satisfies every condition of the chain, so the conditions are consistent and nature's geometry is a model of them.

X. Summary

The passage from the axioms to the algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is a chain of seven steps. Axiom III, read on the summands, gives three carriers of three characters. The conjugacy clause of Axiom I, read as complex type, makes the quaternions the immanent carrier and the other two $\mathbb{C}$ and $M_n(\mathbb{C})$. In the focus of the two laws $\mathbb{C}$ is omniscient and $M_n(\mathbb{C})$ transcendent. Re-projection fixes $n = 3$, and the closure $c_1^2 + N^2 = H^2$ confirms the rank three with the triangle $\{3, 4, 5\}$.

The gauge group of the Standard Model, $U(1) \times SU(2) \times SU(3)$, is the unitary group of this algebra reduced by unimodularity, with one factor from each carrier. The internal space $\mathbb{C}P^2 \times S^2$ carries the inner symmetries of the algebra as its rotations.

Besides the axioms, the chain rests on its stated identifications: that comparison has the structure of a *-algebra, the faithful complex context of evaluation, the carrier condition, the conjugacy condition, the labeling, re-projection, and, as a second route to the rank, the closure. It rests on

theorems of mathematics (Frobenius, Artin-Wedderburn, and the theorem that a faithful finite representation makes an algebra semisimple) that owe nothing to the IM.

KEY CONCEPTS IN THIS CHAPTER

- **Comparison algebra** ($\mathbf{A_F}$): the algebra of the internal part of comparison, $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.
- **Characters**: commutative ($\mathbb{R}$, $\mathbb{C}$), non-commutative division ($\mathbb{H}$ alone), non-division ($M_n(D)$, $n \geq 2$).
- **Context of evaluation**: the complex Hilbert space on which the observables act faithfully; an identification, from which semisimplicity follows as a theorem (Section III).
- **Carrier condition**: Axiom III read on the summands: one modality per carrier, a carrier for each modality, no two carriers of one character.
- **Conjugacy condition**: the conjugacy clause of Axiom I read as complex type for the omniscient and transcendent carriers; it forces $\mathbb{H}$ as the immanent carrier and the field $\mathbb{C}$.
- **Three-Carrier Theorem**: the two conditions give $\mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C})$, with $\mathbb{H}$ immanent.
- **Labeling**: $\mathbb{C}$ omniscient, $\mathbb{H}$ immanent, $M_n(\mathbb{C})$ transcendent, in the focus of the two laws; exchanging two labels would reverse Axiom II.
- **Re-projection**: the content repeats the structure of comparison: three colors from the three positions of comparison, the modalities.
- **Closure**: $c_1^2 + N^2 = H^2$; with $N = 4$ it gives $c_1 = 3$ and $H = 5$.
- **Ray space**: the space of complex lines in a module; for $\mathbb{H}$ it is the sphere S^2, while its states as a real algebra reduce to one.
- **Unimodularity**: the determinant of the two-sided action over the Hilbert space of the Standard Model is one; it gives $U(1) \times SU(2) \times SU(3)$ and the hypercharges.
- **Identification**: a stated step from the structure of comparison to a condition on the algebra, with the observation that bears on it.

- **Associativity**: comparisons composed in a chain give one result however the chain is grouped; it excludes the octonions and the exceptional Jordan algebra.

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PART III

The Derivation of Quantum Mechanics



Chapter 6

The Hilbert Space from the Comparison Algebra

The GNS Construction, Read as the Root Tautology

ABSTRACT

This chapter shows that the Hilbert space of quantum mechanics is derived, not postulated, via the GNS construction applied to the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The GNS construction can be read as the Root Tautology in algebraic form: an algebra is faithfully carried by its states, as an object is determined by its morphisms (the Yoneda lemma), as interaction is comparison. The sector-by-sector computation yields the smallest faithful representation, $H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 = \mathbb{C}^6$; the finite geometry of the Standard Model uses a larger representation of the same algebra, $\mathbb{C}^{96}$, on which the gauge group $U(1) \times SU(2) \times SU(3)$ and the hypercharges are computed. The same construction, applied to the functions on spacetime, gives $L^2(M)$, and with the spinor module of the spacetime Dirac operator the space $L^2(M, S)$ of spinor fields, and the full Hilbert space of quantum mechanics is $L^2(M, S) \otimes H_F$. The first four postulates of quantum mechanics (Hilbert space structure, observables as self-adjoint operators, unitary evolution, and tensor products for composite systems) are derived as consequences. Momentum, the generator of translations of the spacetime factor, gives the canonical commutator $[x, p] = i\hbar$. Superposition and entanglement emerge as structural features of the representation, not as additional postulates. The fermionic term of the spectral action gives the Dirac equation, and the real structure of the triple gives the antiparticles it requires.

I. The Root Tautology in Three Languages

The passage from an algebra to a Hilbert space is not a technical maneuver. It can be read as the mathematical form of the first principle of the Immanent Metaphysics, the Root Tautology: comparison, interaction and relation are one act, named three ways.

The principle has a counterpart in two mathematical languages.

In the language of the Immanent Metaphysics: interaction is comparison is relation. This cannot be denied without self-contradiction, because any denial of the identity compares the tautology with something and finds them different, and so uses the structure it denies.

In the language of category theory: an object is determined by its morphisms. That an object is determined by its morphisms is the content of the Yoneda lemma. By the Yoneda lemma, two objects are isomorphic if and only if the two objects have the same pattern of morphisms. *Category Theory and the Immanent Metaphysics* identifies the two as one content in two languages: "The Root Tautology and the Yoneda Lemma have the same content: an entity is determined by its relationships. They differ in ground." The tautology is grounded in the self-defeat of its denial; the lemma is grounded in the axioms of category theory, and its denial is a mathematical error, not a performative self-contradiction.

In the language of operator algebras: an algebra is faithfully carried by the totality of its evaluations. That an algebra is faithfully carried by the totality of its evaluations is the content of the GNS construction. In the GNS construction, every state of a C*-algebra determines a Hilbert space and a representation, and the direct sum of the representations of all the states of the C*-algebra is faithful. *Non-Commutative Geometry and the Immanent Metaphysics* reads this in parallel with the Yoneda lemma, and says of the parallel: "The parallel is a reading, not an identity of theorems, and no proof in this book depends on it."

Immanent Metaphysics	**Category Theory**	**Operator Algebras**
Interaction = comparison = relation	Object determined by morphisms	Algebra carried by its states
The Root Tautology	The Yoneda Lemma	The GNS Construction

What does the reading mean for the derivation of quantum mechanics? It means that the passage from the comparison algebra A_F to a Hilbert space H_F adds nothing foreign to the algebra. The Hilbert space is built from the algebra's own states, its own ways of being evaluated, as comparison determines its own outcomes. No step of the derivation rests on the reading; the GNS construction is a theorem, and the Hilbert space it yields is what the postulates of quantum mechanics need.

> **Method note: self-reference without paradox.**
>
> The Root Tautology applies to itself: the act of denying it is itself a comparison. The categorical counterpart of that self-application is cartesian closure: the morphisms between two objects form an object of the same kind, to which the Yoneda lemma applies as it applies to the objects themselves, with no regress. The lemma is a theorem and not a fixed point; what it shares with the tautology is its content, and what it lacks is the tautology's ground, which no formal system can capture together with its content.

II. The GNS Construction

The GNS theorem (Gel'fand-Naimark 1943, Segal 1947) states the following. Let A be a C*-algebra and φ a state on A, that is, a positive linear functional with $\varphi(1) = 1$. Then there exists a triple $(H_\varphi, \pi_\varphi, \Omega_\varphi)$ consisting of:

a Hilbert space H_φ,

a *-representation $\pi_\varphi: A \to B(H_\varphi)$ (a *-homomorphism from A to the algebra of bounded operators on H_φ),

a cyclic vector $\Omega_\varphi \in H_\varphi$ such that $\pi_\varphi(A)\Omega_\varphi$ is dense in H_φ,

satisfying the fundamental equation:

$$\varphi(a) = \langle \Omega_\varphi, \pi_\varphi(a)\, \Omega_\varphi \rangle$$

for all $a \in A$. The triple is unique up to unitary equivalence.

The construction

The Hilbert space is built as follows. Define an inner product on A itself:

$$\langle a, b\rangle_\varphi = \varphi(a^*b)$$

The inner product is sesquilinear, positive semidefinite, and conjugate-symmetric. It may fail to be positive definite: there may be elements $a \neq 0$ with $\varphi(a^*a) = 0$. These form the left kernel $N_\varphi = \{a \in A : \varphi(a^*a) = 0\}$, which is a left ideal of A.

The quotient A/N_φ has a well-defined positive definite inner product: $\langle [a], [b]\rangle = \varphi(a^*b)$, where $[a]$ denotes the equivalence class. The Hilbert space H_φ is the completion of A/N_φ with respect to this inner product.

The representation π_φ is left multiplication: $\pi_\varphi(a)[b] = [ab]$. The cyclic vector is $\Omega_\varphi = [1]$, the equivalence class of the identity.

The IM reading

In the language of the Immanent Metaphysics: given the structure of comparison (A) and a specific act of comparison (φ), the act determines a space of outcomes (H_φ), a way of encoding the comparison's content as operations on that space (π_φ), and a reference point from which the comparison is performed (Ω_φ). The reference point from which a comparison is performed is its subjective side (Chapter 3, Section II), so every Hilbert space the construction builds carries the division between the side that compares and the side compared, the cut of Chapter 9, Section III. The account of the GNS construction in the language of the Immanent Metaphysics is a reading, as the dictionary of Section I is a reading.

The inner product $\langle a, b\rangle_\varphi = \varphi(a^*b)$ is the measure of how similar two outcomes are, which is itself a comparison. The fundamental equation $\varphi(a) = \langle \Omega_\varphi, \pi_\varphi(a)\ \Omega_\varphi\rangle$ says that the state's evaluation of any element

equals the inner product of the cyclic vector with the element's action on that vector. In IM terms: the result of a comparison is fully determined by the comparison's reference point and the operation compared.

III. The GNS on A_F: Sector by Sector

Since $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is a direct sum, the construction decomposes summand by summand. A pure state of A_F lives on one summand only; to obtain all three at once, the construction is applied to a state that is a weighted sum of one pure state on each summand, and its Hilbert space is the direct sum of the three irreducible modules. The algebras are real, so the construction is made on their complex span in the context of evaluation of Chapter 5, Section III, and each summand acts on its smallest faithful complex module.

The $\mathbb{C}$ sector (omniscient)

$\mathbb{C}$ acts on $\mathbb{C}^1$. The space is one-dimensional, with no internal structure and no choice of basis. As a real algebra $\mathbb{C}$ has a single state, and its only observables are real numbers: an invariant label, laid out as structure. Its unit group is the circle of phases, and in the Standard Model it is the source of the $U(1)$ of hypercharge.

The $\mathbb{H}$ sector (immanent)

$\mathbb{H}$ acts on $\mathbb{C}^2$, the unit quaternion $q = a + bi + cj + dk$ acting as the matrix with rows $(a + bi, c + di)$ and $(-c + di, a - bi)$. The space is two-dimensional: the qubit space. The unit quaternions form $SU(2)$. As a real algebra $\mathbb{H}$ has only real numbers as self-adjoint elements, since the self-adjoint quaternions are the real scalars; its non-commutativity shows in its unitaries, not in its self-adjoint elements. The complex lines in $\mathbb{C}^2$ form its ray space, the sphere $\mathbb{C}P^1 = S^2$, on which the unit quaternions act by ro-

tations: the space of all orientations of a two-level comparison. The operators on $\mathbb{C}^2$ that $\mathbb{H}$ generates over $\mathbb{C}$ are all of $M_2(\mathbb{C})$, and their self-adjoint operators, the Pauli matrices among them, are the observables of a qubit.

In the Standard Model this sector is weak isospin. The same group, the unit quaternions, acting on the spinor fields of spacetime, gives spin (Chapter 8, Section IV).

The $M_3(\mathbb{C})$ sector (transcendent)

$M_3(\mathbb{C})$ acts on $\mathbb{C}^3$. Its pure states are the rays of $\mathbb{C}^3$, which form the complex projective plane $\mathbb{C}P^2$: a whole space of possible states, the range from which a finding selects one. Its unitaries form $U(3)$, and in the Standard Model it is color.

The smallest faithful representation

The direct sum of the three modules is

$$H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 = \mathbb{C}^6,$$

the smallest faithful representation of A_F. It is enough for the postulates of quantum mechanics, which need a Hilbert space on which the observables act faithfully. The unitary group of A_F acts on it as $U(1) \times SU(2) \times U(3)$.

The Hilbert space of the Standard Model

The finite geometry of the Standard Model uses a larger representation of the same algebra, in which each carrier can act on a state from the left or from the right. Its states fall into sectors, each named by the carrier acting on the left and the carrier acting on the right, and every two carriers share a sector: the lepton doublet ($\mathbb{H}$ on the left, $\mathbb{C}$ on the right), the quark doublet in three colors ($\mathbb{H}$ and $M_3(\mathbb{C})$), the right-handed leptons ($\mathbb{C}$ and $\mathbb{C}$), and the right-handed quarks in three colors ($\mathbb{C}$ and $M_3(\mathbb{C})$). Together, the four sectors hold sixteen particle states per generation. The

real structure J, the antilinear map of the triple that carries each particle state to its antiparticle, exchanges the sixteen particle states with sixteen antiparticle states:

$$\mathcal{H}_F = \mathbb{C}^{2 \times 16 \times 3} = \mathbb{C}^{96}$$

for three generations (Chamseddine, Connes and Marcolli, 2007; *Non-Commutative Geometry and the Immanent Metaphysics*, Chapters 8 and 10). Every two carriers share a sector. Read as a condition on any representation, this linking carries the inseparability of the modalities to the algebra (an identification; *Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 8); the Standard Model's representation satisfies it, and under it the three carriers form a single cycle, the form Axiom II gives the modalities.

On $\mathcal{H}_F$ a unitary acts from both sides of each state at once. The unimodularity condition of Chapter 5, taken over the whole of $\mathcal{H}_F$, reduces the unitary group to $U(1) \times SU(2) \times SU(3)$, up to a finite group, and the phases of that action on each sector are the hypercharges of the Standard Model: $1/6$ for the quark doublet, $2/3$ and $-1/3$ for the right-handed up and down quarks, $-1/2$ for the lepton doublet, -1 for the right-handed electron, and 0 for the right-handed neutrino. They are computed from the representation, not put in.

The comparison triple (a spectral triple, that is, an algebra, a Hilbert space on which it acts and a Dirac operator, together with the real structure J and a grading γ that splits states into left-handed and right-handed, satisfying the conditions of *Non-Commutative Geometry and the Immanent Metaphysics*, Definition 4.1) does not fix this representation in full: the theorem of *Non-Commutative Geometry and the Immanent Metaphysics* "does not determine the representation in full or the entries of D". What it proves is that every algebra satisfying its conditions is A_F, and that the Standard Model's own finite geometry, on $\mathbb{C}^{96}$, satisfies every one of them.

IV. The Three-Column Dictionary for Hilbert Space

The construction extends the dictionary to the Hilbert space structure. Each row pairs a concept of the Immanent Metaphysics with what carries it in quantum mechanics, in category theory and in operator algebra. The pairings are readings, and each formal column has its theorems in its own setting.

Immanent Metaphysics	**Quantum mechanics**	**Category theory**	**Operator algebra**
Comparison	Measurement	A cohesive topos	A comparison triple $(A, \mathscr{H}, D, J, \gamma)$
Sameness and difference	Outcomes found equal or unequal	Points identified or distinguished	The Dirac operator D; spectral distance
Subjective and objective	Apparatus and system	The two sides of an adjunction	The *-involution, $(ab)^* = b^*a^*$
Content and context	Eigenvalue and setting	The point a probe places; the position in the probe	Values of observables and the states they are evaluated in; the grading γ
Symmetry	Exact invariance	Every probe constant (discrete)	A central element
Continuity	No jump within one setting	Every assignment of points a probe (codiscrete)	An element commuting with D
Incommensuration Theorem	The uncertainty principle, an instance	No object with two or more points is both discrete and codiscrete	A symmetric continuous observable is constant on each class of comparison

The concordance continues with the modalities, labeled in the focus of the two laws (Chapter 5, Section VI), with the two modes and with the qubit.

Immanent Metaphysics	**Quantum mechanics**	**Category theory**	**Operator algebra**
Omniscient	U(1) of hypercharge	$\flat = \mathrm{Disc} \circ \Gamma$	The center Z(A); the carrier $\mathbb{C}$
Immanent	SU(2) of weak isospin (and, in spacetime, spin)	The object itself	A itself; the carrier $\mathbb{H}$
Transcendent	SU(3) of color	$\sharp = \mathrm{CoDisc} \circ \Gamma$	$B(\mathscr{H})$; the carrier $M_3(\mathbb{C})$
CA mode	Spacetime	Codiscrete objects	The manifold factor of M × F
SD mode	Discrete, exact findings	Discrete objects	The classes of comparison of the finite factor
The qubit	$\mathbb{C}^2$ with its Bloch sphere	(none)	The module of $\mathbb{H}$; its ray space S^2

In the category-theory column, a cohesive topos is a universe of spaces joined to the category of sets by a chain of adjunctions (Chapter 12, Section III); an adjunction is a pair of opposed maps between two categories, each the best approximation to an inverse of the other; $\flat = \mathrm{Disc} \circ \Gamma$ keeps only the points of an object, with no cohesion among them, which makes it discrete; and $\sharp = \mathrm{CoDisc} \circ \Gamma$ joins all its points into one indiscrete whole, which makes it codiscrete.

The dictionary is a concordance: each concept occupies one row, and each carrier appears once. No functor carries one formal column to another; the center of an algebra, for instance, is not carried along by its homo-

morphisms. The concordance is a check, since each concept has received independent formalizations, and in each formalization the same theorem holds in the terms of that formalization.

V. From Finite to Infinite Dimensions

The finite Hilbert space encodes the internal degrees of freedom: color, isospin, hypercharge. But quantum mechanics operates on an infinite-dimensional Hilbert space. Position, momentum, orbital angular momentum, and the full apparatus of continuous observables require infinitely many dimensions.

The infinite-dimensional structure comes from the CA mode: the continuous, asymmetric mode that the ICT identifies with external spacetime. The algebra of the CA mode is the commutative algebra $C^\infty(M)$ of smooth functions on a spacetime manifold M. The representation of that algebra on $L^2(M, S)$, the space of square-integrable spinor fields on M, provides:

infinite-dimensional structure (infinitely many independent spinor fields),

position and momentum operators (multiplication and differentiation),

the spacetime Dirac operator D_M (which encodes the metric of spacetime).

The infinite-dimensional factor is built by the same construction as the finite one. The algebra of the arena is commutative: the continuous functions on M, the completion of $C^\infty(M)$. One faithful state on it is integration against a smooth positive density of total weight one. The GNS construction on that state gives $L^2(M)$, the square-integrable functions on M, with the algebra acting by multiplication; two such densities give unitarily equivalent spaces, by multiplication with the square root of their ratio. The spinor fields are the sections of the spinor bundle S, a module over the functions on M on which the Dirac operator acts, and $L^2(M, S)$ is its

completion: $L^2(M)$ with the spinor fiber at each point. Connes' reconstruction theorem shows that a commutative spectral triple with real structure meeting his conditions is the spin geometry of a compact Riemannian manifold (Connes, 2013); the arena is taken in that form, as the spectral action takes it (Section VI). So the space of wave functions on spacetime is built from the algebra of the arena by the GNS construction and the spinor module of its Dirac operator, as H_F is built from the algebra of the internal part.

The full Hilbert space is the tensor product

$$H = L^2(M, S) \otimes H_F,$$

the product of spacetime and internal degrees of freedom. Every quantum state is a spinor field on spacetime tensored with an internal state, or a sum of such products. The tensor product reflects the ICT's two surviving modes, composed side by side in the product $M \times F$.

Position, momentum and their commutator

Momentum is the generator of translations of the spacetime factor. In a local flat chart, the translations by a distance a along one coordinate of space form a continuous one-parameter group of unitary operators $U(a)$ on $L^2(M, S)$, with $U(a)U(b) = U(a + b)$ and $U(0) = 1$. By Stone's theorem, as for Postulate 3 in Section VI, $U(a) = \exp(-ipa/\hbar)$, with p a self-adjoint operator: the momentum along that coordinate. Position x is the operator of multiplication by the same coordinate: position is a coordinate of the continuum. Translating and then reading position adds a to the position: $U(a)^* \, x \, U(a) = x + a$. Differentiating at $a = 0$ gives $(i/\hbar)[p, x] = 1$, that is,

$$[x, p] = i\hbar.$$

With the Robertson inequality, $\Delta x \, \Delta p \geq \tfrac{1}{2}|\langle [x, p] \rangle|$, this gives the bound $\Delta x \, \Delta p \geq \hbar/2$: the uncertainty principle, which Chapter 4, Section III states as an instance of the Incommensuration Theorem. The constant $\hbar$

sets the unit in which momentum is measured. That action comes in a least step at all is shown in *Measurement*, Chapter 9, Section V, which identifies the least step of intrinsic circulation with ħ (Chapter 8, Section IV).

The spacetime factor is the subject of *The Gravitational Passage*. *Measurement* derives one dimension of time against three of space (its Chapter 11, Section IX), and the Lorentzian form of the product geometry exists (Barrett, 2007). The internal factor is a finite space: it has the sphere S^2 and the plane $\mathbb{C}P^2$ only as the ray space and the pure state space of two of its carriers, with no geometry of its own beyond them. *The Gravitational Passage* gives the same pair of spaces radii. The two accounts are two instances of one schema, read at their own scopes: on the side of content, as here, a space of states with no size; on the side of context, as there, a geometry with radii.

The full spectral triple

For the Standard Model, the product spectral triple is

$$(C^\infty(M) \otimes A_F,\ L^2(M, S) \otimes \mathscr{H}_F,\ D_M \otimes 1 + \gamma_5 \otimes D_F),$$

with the real structure $J = J_M \otimes J_F$ and the grading $\gamma = \gamma_5 \otimes \gamma_F$. Here $\mathscr{H}_F$ is the 96-dimensional space of Section III, and D_F is the finite Dirac operator on it, made of the Yukawa matrices that couple left-handed doublets to right-handed singlets (the couplings, through the Higgs field, that give the fermions their masses and mixings). The finite part has KO-dimension 6 (a number modulo 8 fixed by the signs relating J, γ and D; Chapter 8) and the manifold 4, so the product has KO-dimension 10, which is 2 modulo 8.

This spectral triple encodes the Standard Model coupled to gravity. Its algebra is derived: every comparison triple satisfying the conditions of Chapter 5 has the algebra A_F, and the Standard Model's own triple is one of them (*Non-Commutative Geometry and the Immanent Metaphysics*, Theor-

ems 12.1 and 12.2). The numerical entries of D_F, the masses and mixings, are not fixed by the derivation, and I name those entries as the place where the derivation stops.

VI. What This Layer Derives

The GNS construction on A_F, combined with the spectral triple formalism, derives four of the six standard postulates of quantum mechanics. Each derivation is traced to its structural origin.

Postulate 1: Hilbert space

Standard QM: The state of a quantum system is described by a vector in a Hilbert space.

IM derivation: The Hilbert space H_F is the GNS representation space of A_F. It is derived, not postulated, via the chain: Root Tautology → six intrinsics → three modalities → ICT → three axioms → $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ → GNS construction → $H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3$. The full Hilbert space is $H = L^2(M, S) \otimes H_F$, whose first factor is built from the algebra of the arena by the GNS construction and the spinor module of its Dirac operator (Section V).

Postulate 2: observables

Standard QM: Observables are self-adjoint operators on the Hilbert space.

IM derivation: Observables are the self-adjoint operators of the complex algebra that A_F generates on H_F, $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$; for the quaternions this adds the Pauli matrices, which the real algebra $\mathbb{H}$ does not contain as observables. The *-involution on A_F, which encodes the reversal of comparison (the exchange of subjective and objective), becomes

the adjoint on $B(H_F)$. The spectral theorem guarantees that every self-adjoint operator has a real spectral decomposition: measurement outcomes are real numbers.

Postulate 3: unitary evolution

Standard QM: Time evolution is governed by a unitary operator $U(t) = \exp(-iHt/\hbar)$.

IM derivation. The postulate says two things: that evolution is unitary, and that a Hamiltonian H generates it. The first comes from what evolution between comparisons is; the second from the dynamics of the spectral triple.

Evolution between comparisons preserves the structure of comparison. Evolution time is the transcendent time of Chapter 7: the time of continuous, reversible change in what is possible, during which no comparison is made and nothing is found. A change in which nothing is found registers no difference, so it erases no distinction that the algebra of comparison can draw: two observables that differ before it still differ after it. It respects the operations of the algebra, since composing two comparisons and then evolving is evolving and then composing, and it respects the involution, since exchanging the one who compares with what is compared does not depend on when the exchange is read. So evolution over a time t is a *-automorphism α_t of the algebra of observables. Evolution over s and then over t is evolution over $s + t$, and evolution over no time changes nothing: $\alpha_{s+t} = \alpha_s \circ \alpha_t$ and $\alpha_0 = \mathrm{id}$. Evolution time is the parameter of the formalism, and the formalism that describes content is continuous (*Measurement*, Proposition 8.1 and Table 12.1), so α_t depends continuously on t. This step is an identification: evolution between comparisons is identified with a continuous one-parameter group of automorphisms of the algebra of comparison. The observation that bears on it is that closed systems evolve reversibly, and that what an isolated system can be found to be is never lost by its evolution alone.

Every such group is unitary. Two theorems finish the step. Every automorphism of the algebra of all operators on a Hilbert space that preserves the structure of comparison is implemented by a unitary operator, $a \mapsto u\, a\, u^*$ (in finite dimensions this is the Skolem-Noether theorem; in general it is the theorem of Wigner and Kadison). A continuous one-parameter group stays in the part of the group of automorphisms connected to the identity, where every automorphism is inner; so complex conjugation, which is an automorphism of the real algebras $\mathbb{C}$ and $M_3(\mathbb{C})$ and is implemented antiunitarily, never enters an evolution. For a continuous one-parameter group, the unitaries can be chosen as a continuous group $U(t)$, and Stone's theorem says that every such group has the form

$$U(t) = \exp(-iHt/\hbar),$$

with H a self-adjoint operator, unique up to an added constant. States then evolve as $\psi(t) = U(t)\psi$. Evolution by the unitary group $U(t)$ is the form of Postulate 3. The constant $\hbar$ sets the unit in which H is measured.

What the generator keeps. The structure of comparison is the same at every moment, so an evolution over a given time is the same whenever that time begins; that is the group law $U(s)U(t) = U(s + t)$. The generator H commutes with every $U(t)$, so the evolution keeps it: H is the energy, the quantity that this sameness from moment to moment keeps. A state of definite energy E therefore keeps its form in time, while its phase advances at the steady rate $E/\hbar$. The translations in space form a group of the same kind (Section V), and its generator is the momentum p, which the evolution keeps wherever the structure is the same at every place; the phase of a state of definite momentum varies across space at the rate $p/\hbar$. The advance of phase in time and its variation across space together fix how a state changes between comparisons, given its energy and its momentum.

Which Hamiltonian. The generator H is the dynamics, and the spectral triple fixes it. The spectral action on the product triple,

$$S = \operatorname{Tr} f(D_A/\Lambda) + \tfrac{1}{2}\langle J\xi, D_A\xi\rangle,$$

in which D_A is the Dirac operator with the gauge fields added, ξ a fermion field, f a cutoff function and Λ the energy scale at which the action is evaluated, yields, through its bosonic term, the Einstein-Hilbert action together with the Yang-Mills and Higgs actions, and, through its fermionic term, the couplings of the fermions, among them the mass terms carried by the finite Dirac operator D_F (Chamseddine, Connes and Marcolli, 2007). The Hamiltonian is the generator of time translations of that action. The action is evaluated on a Riemannian manifold, and the Lorentzian form of the product geometry exists (Barrett, 2007).

Each state's own time. Operator algebra adds one more fact, which the Root Tautology reads. A faithful state on an algebra of observables determines, with no further data, a one-parameter group of automorphisms of the algebra, its modular flow (Tomita and Takesaki). In finite dimensions, for the state $a \mapsto \mathrm{Tr}(\rho a)$ with ρ invertible, the modular flow is $a \mapsto \rho^{it}\, a\, \rho^{-it}$: a unitary evolution generated by $-\log \rho$. The state is unchanged by its own flow, and it satisfies with respect to that flow the condition of thermal equilibrium (the KMS condition); for an equilibrium state $\rho = e^{-\beta H}/Z$ the modular flow is the evolution generated by H itself, with the time measured in units of $\hbar\beta$. The modular flows of two faithful states differ only by inner automorphisms (Connes), so for the algebras of quantum field theory the flow defines a class of time evolutions that depends on the algebra alone. Connes and Rovelli proposed that physical time is the modular flow of the physical state. In the terms of the Immanent Metaphysics, an act of comparison, carried by a faithful state, carries its own time, and that time is unitary by construction. This reading is an identification, and no step of the derivation rests on it; it shows that the unitary form of evolution is already present in the algebra of comparison and its states.

Postulate 4: tensor products

Standard QM: The state space of a composite system is the tensor product of the component state spaces.

IM derivation: Two systems are two comparisons that can be performed together without disturbing each other: their observables commute, and together they generate every observable of the joint system. Reading two systems as two comparisons performed together without disturbing each other is an identification. The two parts of a joint comparison are independent, since neither part disturbs the other, and inseparable, since together the two parts exhaust the joint comparison. The observation that bears on it: two systems measured together, in either order, give the same joint statistics.

A theorem then fixes the composition: if two commuting algebras, each a full matrix algebra, together generate all operators on a finite-dimensional Hilbert space, that space is the tensor product of their modules, and the joint algebra is their tensor product. A direct sum would make every observable of one system act as zero on the other, so the two could not be compared together. The complex algebra of A_F is a direct sum of full matrix algebras, so the theorem applies sector by sector, and the joint algebra of two systems is $A_F \otimes A_F$, acting on $H_F \otimes H_F$. A state of the composite is entangled when it is not a convex combination of product states; a pure state is entangled when it is not a product $|\alpha\rangle \otimes |\beta\rangle$. The tensor product is not an independent postulate; it is a consequence of what a joint comparison is. Chapter 8, Section II treats entanglement as a phenomenon of comparison: a joint comparison whose finding does not decompose into comparisons of the parts.

> **Superposition.**
>
> Superposition is not a separate postulate: it is a consequence of the vector space structure of H_F. A general state is a complex linear combination of basis vectors. In IM terms: within a sector, a system can be in a superposition of states; the comparison relation does not force a definite outcome until a specific comparison (measurement) is performed. Which superpositions can be found is settled by the law of symmetry (Chapter 8, Section I).

The Running Example: The Atom's Spin as a Qubit

The spin of the silver atom of Chapter 3 is a ray of $\mathbb{C}^2$, the space on which the unit quaternions act as spin (Section VII; Chapter 8, Section IV), and so a point of the Bloch sphere S^2. "Up" and "down" along the axis of the magnet are the two poles, and the states between them are the other points of the sphere. The observable the magnet measures is a Pauli matrix, a self-adjoint operator of the complex algebra $M_2(\mathbb{C})$ that $\mathbb{H}$ generates on its module; its two eigenvalues are the two spots on the screen. The unit quaternions turn the sphere, which is what turning the magnet does to the question asked.

VII. The Dirac Equation and Antimatter

The postulates of Chapter 1 are those of non-relativistic quantum mechanics. The product triple of Section V carries the relativistic equation of the electron as well. The fermionic term of the spectral action, $\frac{1}{2}\langle J\xi, D_A\xi\rangle$, has as its equation of motion $D_A\xi = 0$. On the product geometry, D_A is $D_M \otimes 1 + \gamma_5 \otimes D_F$ with the gauge fields added, so this is the Dirac equation of each fermion: the spacetime Dirac operator, coupled to the gauge fields of the three carriers, with the masses and mixings carried by the entries of D_F through the Higgs field (Chamseddine, Connes and Marcolli, 2007). In the Lorentzian form of the product geometry (Barrett, 2007) it is the Dirac equation in its familiar form, $(i\gamma^\mu D_\mu - m)\psi = 0$, for each fermion of mass m (Dirac, 1928).

The real structure J of the triple carries each particle state to its antiparticle state: on $\mathscr{H}_F$ it exchanges the sixteen particle states of each generation with the sixteen antiparticle states, and on states it is charge conjugation (Chapter 8, Section VI). The comparison triple includes J by definition (*Non-Commutative Geometry and the Immanent Metaphysics*, Definition 4.1), so every particle state comes with its antiparticle state, as Dirac's

equation requires and as the positron, found in 1932, confirmed (Anderson, 1933). Chapter 8, Section VI reads J as the exchange of the two terms of a state, toward the thing compared against and toward the thing compared. The two terms of a state are the transcendent and the omniscient, which Axiom I makes conjugate. So the real structure carries the conjugacy of Axiom I to the states, and antimatter is the conjugacy of Axiom I made visible.

The unit quaternions, which carry the immanent modality in the focus of the two laws, form the group $SU(2)$. $SU(2)$ acts twice in physics: on the $\mathbb{H}$ factor of A_F as weak isospin, and on the spinor fields of $L^2(M, S)$ as spin, the double cover of the rotations of space. The spinors on which the Dirac equation acts are the second of these; Chapter 8, Section IV states the two actions, why they are one group, and the Bloch sphere S^2 on which both act.

VIII. The Born Rule: A Preview

The fundamental equation of the GNS construction, applied to a projection P, gives $\varphi(P) = \langle \Omega_\varphi, \pi_\varphi(P)\Omega_\varphi \rangle = \|\pi_\varphi(P)\Omega_\varphi\|^2$, the form in which the Born weight is written. Chapter 7 shows why the value of a state on a projection is the probability of the finding, and its Section V sets out what the GNS construction supplies.

IX. Summary

The Hilbert space of quantum mechanics is not a postulate. It is the GNS representation space of the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$: the unique algebra derived in Chapter 5 from the structure of comparison, through the theorem, the three axioms and the identifications that chapter states.

The passage from algebra to Hilbert space can be read as the Root Tautology in algebraic form: an algebra is faithfully carried by its states, as interaction is comparison, as an object is determined by its morphisms. The GNS construction does not add structure to the comparison algebra. It builds the space of outcomes from the algebra's own states.

The three sectors of the smallest faithful representation ($\mathbb{C}^1$, $\mathbb{C}^2$, $\mathbb{C}^3$) carry the three modalities: omniscient ($\mathbb{C}$, the source of the $U(1)$ of hypercharge), immanent ($\mathbb{H}$, $SU(2)$), transcendent ($M_3(\mathbb{C})$, color). On the Hilbert space of the Standard Model, $\mathbb{C}^{96}$, the unitary group of the algebra, reduced by unimodularity, is the gauge group $U(1) \times SU(2) \times SU(3)$, and its phases are the hypercharges.

What this means for the physicist: the Hilbert space of quantum mechanics, the gauge group of the Standard Model (given its representation and the unimodularity condition, stated as an input), the spin-1/2 property, and the superposition principle are not independent postulates assembled from experimental hints. They are structural consequences of a single chain of derivation that begins with the tautological observation that interaction is comparison. Postulates 3 and 4 each rest on one stated identification: evolution between comparisons as a continuous group of automorphisms, and two systems as two comparisons performed together without disturbing each other. The canonical commutator $[x, p] = i\hbar$ follows from momentum as the generator of translations. The space of spinor fields on spacetime is built by the same GNS construction as the finite space, with the spinor module of the spacetime Dirac operator, and the Dirac equation and its antiparticles come from the spectral action and the real structure of the same triple.

The chain is: comparison → algebra → states → Hilbert space. The first arrow is the chain of Chapter 5, and the other two arrows are theorems. The steps contain no continuous parameter; what they leave open is named, the entries of D_F. The Hilbert space of quantum mechanics is the space that comparison builds for itself.

KEY CONCEPTS IN THIS CHAPTER

- **GNS construction**: The Gel'fand-Naimark-Segal theorem: given a C*-algebra A and a state φ, the construction yields a Hilbert space H_φ, a representation π_φ, and a cyclic vector Ω_φ. Read as the Root Tautology in the three-column dictionary. Applied to the functions on spacetime, it gives $L^2(M)$, and with the spinor module of the Dirac operator, $L^2(M, S)$ (Section V).
- **Root Tautology**: The principle that interaction = comparison = relation. It cannot be denied without self-contradiction. The Yoneda lemma has the same content with a different ground; the GNS construction is read in parallel.
- $\mathbf{H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3}$: The finite Hilbert space obtained by applying the GNS construction sector-by-sector to A_F. Total dimension 6, the smallest faithful representation.
- $\boldsymbol{\mathcal{H}_F = \mathbb{C}^{96}}$: The Hilbert space of the finite geometry of the Standard Model: sixteen particle and sixteen antiparticle states per generation, three generations; the gauge group and the hypercharges are computed on it.
- **Spectral triple**: The data (A, H, D): an algebra, a Hilbert space on which it acts, and a Dirac operator; with a real structure J and a grading γ it is the comparison triple of Section III. It encodes both spacetime and internal degrees of freedom. The full spectral triple is $(C^\infty(M) \otimes A_F, L^2(M, S) \otimes \mathscr{H}_F, D_M \otimes 1 + \gamma_5 \otimes D_F)$, with real structure J and grading γ.
- **Spectral action**: The action principle $S = \mathrm{Tr}\, f(D_A/\Lambda) + \frac{1}{2}\langle J\xi, D_A\xi\rangle$ that derives the Standard Model Lagrangian plus Einstein gravity from the spectral triple. It supplies the generator of evolution; the unitary form comes from evolution as a continuous group of automorphisms (Postulate 3).

- **Modular flow**: The one-parameter group of automorphisms that a faithful state determines by itself (Tomita and Takesaki); for an equilibrium state it is the evolution generated by the Hamiltonian; read in the IM as the time an act of comparison carries.
- **Spin and the quaternions**: The unit quaternions form $\mathrm{SU}(2)$, which acts on the $\mathbb{H}$ factor as weak isospin and on the spinor fields of spacetime as spin. In the focus of the two laws, the immanent modality's quaternionic structure appears once in each mode of the ICT (Chapter 8, Section IV).
- **Composite systems**: Two comparisons performed together without disturbing each other (an identification); their joint algebra is the tensor product, and a state not a convex combination of product states is entangled (Postulate 4).
- **Canonical commutator**: $[x, p] = i\hbar$, from momentum as the generator of translations and Stone's theorem (Section V).
- **Dirac equation**: The equation of motion $D_A\xi = 0$ of the fermionic term of the spectral action; in Lorentzian form, $(i\gamma^\mu D_\mu - m)\psi = 0$ (Section VII).
- **Antimatter**: The antiparticle states that the real structure J pairs with every particle state; J carries the conjugacy of Axiom I to the states (Section VII).
- **Three-column dictionary**: A concordance recording each concept of the IM with what carries it in quantum mechanics, category theory and operator algebra. No functor joins the formal columns; each has its theorems in its own setting.

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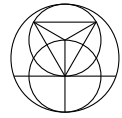
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Chapter 7

The Born Rule from the Structure of Comparison

Why Probability, Why This Probability, and Why Nothing Else

ABSTRACT

The Born rule, the prescription that the probability of obtaining a measurement outcome k is $p(k) = |\langle\psi|e_k\rangle|^2$, is the most mysterious of the quantum mechanical postulates. It connects the mathematical formalism to the experimental world, and yet no textbook derives it from deeper principles. This chapter presents an argument that traces the Born rule to the structure of comparison itself. The argument proceeds in three stages. First, the Incommensuration Theorem establishes that the state does not fix which outcome a measurement finds, and sameness applied to repetition gives each outcome a stable weight, its probability. Second, Gleason's theorem, applied to the finite sectors and to the full space, with Busch's theorem in the two-dimensional sector, identifies the Born rule as the unique probability measure compatible with the Hilbert-space structure derived in preceding chapters. Third, the two premises that Gleason's theorem requires are shown to follow from the structure of comparison: that the weight of a finding depends only on the subspace that represents it, from the law of symmetry, and that the weights of exclusive outcomes add, by two routes: from the content/context partition and the sameness/difference pair, and from the counting of repeated outcomes. Gleason's theorem in turn returns the residual of the first stage as a theorem of the Hilbert space, in the form of the Kochen-Specker theorem. Two steps are stated as identifications: probability from repetition, and the single weight of a finding from the law of symmetry. The GNS construction supplies the vector and the inner product in which the weight is written.

I. The Problem

Of the six postulates of quantum mechanics surveyed in Chapter 1, five state the structure of the theory and how a state changes: the Hilbert space, the self-adjoint operators, unitary evolution, tensor-product composition, and the eigenvalue outcomes with the state after an outcome. These five specify what the formalism *is* and what becomes of a state.

The sixth postulate is different. It specifies the connection between the formalism and the world of experiment. It states: if a system is in state $|\psi\rangle$ and a measurement is performed whose eigenstates are $\{|e_k\rangle\}$, then the probability of obtaining outcome k is $p(k) = |\langle\psi|e_k\rangle|^2$.

The sixth postulate is the Born rule, which Max Born proposed in 1926. Since 1926, the Born rule has been derived only from premises that already contain probability or an equivalent assumption (see below).

The Born rule has a peculiar status in the logical architecture of quantum mechanics. The other postulates can be motivated by mathematical elegance, by symmetry arguments, or by analogy with classical physics. The Hilbert space is a natural generalization of the phase space. Self-adjoint operators have real eigenvalues, which is what measurements produce. Unitary evolution preserves probabilities, which is what a consistent dynamics requires.

The Born rule, by contrast, has no such motivation. Why the modulus squared? Why not the modulus to the fourth power, or the modulus itself, or some entirely different functional form? The inner product $\langle\psi|e_k\rangle$ is a complex number; the modulus squared is a particular way of extracting a real non-negative number from a complex amplitude. It is the simplest such extraction, but simplicity is an aesthetic criterion, not a derivation.

The problem becomes sharper when one attempts to derive the Born rule from within the quantum formalism itself. Every such attempt either smuggles in an equivalent assumption (as in the decision-theoretic approaches of Deutsch and Wallace) or requires an additional axiom that is itself in need of justification (as in Zurek's envariance argument). The Born rule resists internal derivation because it is the bridge between the mathematical formalism and the physical world, and bridges cannot be built from one bank alone.

On the status of the Born rule in axiomatic programs.

Hardy's five axioms (2001) derive the Born rule for finite-dimensional systems from a probabilistic-operational framework, but the framework itself assumes probabilities from the start. Chiribella, D'Ariano, and Perinotti (2011) derive it from six principles stated within an operational-probabilistic framework, in which probabilities are primitive. In both cases, the Born rule is not derived from non-probabilistic premises; it is redistributed among the axioms. The contribution of the Immanent Metaphysics is to begin from a non-probabilistic starting point, the structure of comparison, and to show that probability itself, and this specific probability, is a structural consequence.

II. Why Probability Is Necessary

The first stage of the argument establishes that measurement outcomes *must* be probabilistic, that deterministic measurement is structurally impossible within any system whose foundations are given by the comparison relation.

The Incommensuration Theorem, derived in Chapter 4, states that symmetry and continuity cannot both be perfected simultaneously in any comparison. Of the four compound concepts constructed from the six intrinsics, continuity (s, s), symmetry (s, d), discontinuity (d, s) and asymmetry (d, d) (Chapter 4), only two pairings survive: the CA mode (continuous and asymmetric) and the SD mode (symmetric and discontinuous).

A measurement is a specific instance of the comparison relation. The apparatus is the perspective from which the comparison is made, and the system is what is compared; the property of the system that is probed is the content, and the setting of the apparatus is the context (Section IV). What is found of the measured system is discontinuously symmetric (Chapter 4, Section III), and the formalism that describes it is continuous:

the two levels of one theory. The formalism is continuous because possibility passes continuously from one direction of the state space to another: nothing in possibility tells the directions apart except that there are several of them. What is found is one outcome, discrete.

The Incommensuration Theorem leaves a residual in every lawful description. *Measurement* states it: "Symmetry and continuity cannot both be perfected in any comparison, so no lawful description of anything specifies every finding exactly: something is always left over that the law does not fix. I call that remainder the residual." A description that fixed every finding would carry one law into every state everywhere, which is symmetry perfected, and would pass each state into the next with no break, which is continuity perfected, and the theorem forbids the two together (*Measurement*, Chapter 14, Sections I and II). The single outcome of a quantum measurement is the plainest instance of the residual: the state before the measurement, known as fully as the theory allows, gives the weight of each outcome, and the outcome found is not contained in it.

The same result follows from the weights. The state is the record of what a setting makes possible, with no further state beneath it. No further state settles the outcome, since the theorem forbids a description that fixes every finding (*Measurement*, Chapter 14, Sections I and II); and no further state shifts the weights without settling the outcome, since the weight of an outcome is the stable frequency of the same comparison, performed on the same state in the same context (later in this section), and the weight of a finding depends only on the subspace that represents it (Section IV). So the state is the whole record a finding draws on, and an outcome to which the state gives a weight above zero could be found. If one outcome were already settled, by anything, hidden or otherwise, no other outcome could be found, and the state would give a weight above zero to that outcome alone, as it does immediately after a finding (Chapter 9, Section V). So while the state gives more than one outcome a weight above zero, nothing settles the outcome before it is found: the weights say how strongly each outcome could be found, and which one is found is settled only in the finding.

On the internal labels the residual has an exact algebraic form. An observable that recorded the finding deterministically would be both symmetric (central, unchanged by every inner symmetry) and continuous (commuting with the Dirac operator). By the algebraic form of the theorem (Chapter 5, Section II), such an observable is constant on every class of comparison: it can report which class is found, the superselected label, and nothing within the class. So the state fixes which class is found, and within a class the finding is not fixed by the state. On the full space of Chapter 6 the residual is the general one stated above: no observable fixes which outcome of a spin, a position or an energy is found.

The outcome is therefore *structurally* probabilistic. Indeterminacy alone does not give a probability: an outcome the state does not fix could have no stable weight at all. The weight comes from sameness applied to repetition: the same comparison, performed on the same state in the same context, is the same comparison, so its outcomes recur with one stable relative frequency, and that frequency is the probability of the outcome. Taking the stable relative frequency as the probability of the outcome is an identification, and I state that step as an identification. The observation that bears on it: repeated measurements on identically prepared systems give relative frequencies that settle to stable values. The probability is not epistemic, not a reflection of our ignorance of hidden variables. It is ontic in the structural sense: the residual belongs to comparison itself, and no further knowledge of the state removes it. The Born rule is the bridge between the two levels of the theory, from the continuous formalism to the discrete findings (Chapter 9, Section IV).

The three times and measurement.

The same distinction can be stated in the language of three times. Evolution time (the time of Postulate 3) belongs to the transcendent modality: it is the time of continuous, reversible evolution at the level of the formalism, which fixes the weights of findings and not the finding. Measurement time (the time of Postulates 5 and 6) belongs to the immanent modality: it is the time of discrete, irreversible, probabilistic actualization. Geometric time (the time of general relativity) belongs to the omniscient modality. Geometric time is time laid out as structure, the whole history taken at once in the four-dimensional geometry; that role is the omniscient one, though the geometry it lays out is of the continuous mode. Axiom I states that the immanent is more fundamental. The measurement moment is not generated by the formalism of unitary evolution; it is a primitive structural fact of the immanent modality. The two times, evolution and measurement, are structurally distinct and cannot be reduced to each other.

The three times also say why time is not an observable. Pauli observed that no self-adjoint time operator can be conjugate to a Hamiltonian whose energies are bounded below (Pauli, 1933). Evolution time is transcendent: it is the parameter of the formalism, the time across which the weights of findings change, and not a content that a comparison finds. So time is not an observable, and the relation between energy and time is of another kind from the relation between position and momentum. Position and momentum are two descriptions of one content, conjugate under Axiom I (Chapter 4, Section III); the energy and the time of an evolution are a generator and its parameter.

III. Gleason's Theorem

The second stage of the argument identifies the Born rule as the unique probability measure on the Hilbert space that satisfies two conditions: each finding has one weight, whatever complete measurement contains it, and the weights of exclusive findings add.

Gleason's theorem (1957) is one of the deepest results in the foundations of quantum mechanics. It states: for any Hilbert space of dimension three or greater, the only probability measure on the lattice of closed subspaces that is additive on orthogonal subspaces is the Born rule.

More precisely: let H be a Hilbert space with $\dim(H) \geq 3$. Let μ be a function that assigns a non-negative real number to every closed subspace of H, such that $\mu(H) = 1$ and $\mu(V_1 \oplus V_2) = \mu(V_1) + \mu(V_2)$ whenever V_1 and V_2 are orthogonal subspaces (and countably additive over orthogonal families, when H is infinite-dimensional). Then there exists a density operator ρ such that $\mu(V) = \mathrm{tr}(\rho\, P_V)$ for every closed subspace V, where P_V is the projection onto V. That μ is a function of the subspace alone, the same whichever complete measurement contains the subspace, is the first premise of the theorem; additivity is the second.

For a pure state $\rho = |\psi\rangle\langle\psi|$, this reduces to the Born rule: $\mu(\mathrm{span}\{|e\rangle\}) = |\langle\psi|e\rangle|^2$ for every unit vector $|e\rangle$.

The significance of Gleason's theorem for this chapter is that it transforms the problem of deriving the Born rule into a different problem: deriving its two premises. If one can show that the weight of a finding depends only on the subspace that represents it, and that the weights assigned to mutually exclusive measurement outcomes must add, then the Born rule follows as a mathematical theorem, not as a physical postulate.

The theorem applies sector by sector. The observables of the finite part, $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ (Chapter 6, Section VI), are block-diagonal on $H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3$, and no observable connects two sectors, so the findings are

the projections of each sector, not every subspace of $\mathbb{C}^6$. In the $M_3(\mathbb{C})$ sector, of dimension three, Gleason's theorem applies directly; in the $\mathbb{H}$ sector, of dimension two, Busch's theorem applies (below); the $\mathbb{C}$ sector has one outcome, whose weight is fixed. For the algebra as a whole the same conclusion is the generalized Gleason theorem for von Neumann algebras (Christensen, 1982; Yeadon, 1983; Bunce and Wright, 1992), which needs exactly this: Busch's form for any two-dimensional summand.

The same holds on the full Hilbert space of Chapter 6, $H = L^2(M, S) \otimes H_F$, on which the findings of position, momentum, spin and energy are made. The two premises of Section IV are stated for any comparison, not for the internal labels alone, so they hold there unchanged. The space is infinite-dimensional, and Gleason's theorem there asks that the weights add over countable orthogonal families; that is the same premise of additivity, applied to a measurement with countably many exclusive outcomes, as the energy levels of an atom are. Within each superselection sector (Chapter 8, Section I) Gleason's theorem then gives the Born rule for every projection, and the generalized theorem for von Neumann algebras gives it for the algebra as a whole. The finite sectors are the case of the internal labels; the full space is the case of every finding.

The two-dimensional sector.

Gleason's theorem, stated for projections, needs dimension three or more, and the $\mathbb{H}$ sector of A_F gives the two-dimensional space $\mathbb{C}^2$, the qubit. Busch (2003) proved the corresponding result for every dimension, two included, when probability is assigned to effects (the positive operators below the identity, which include every projection and every unsharp measurement) and is additive over the outcomes of a measurement. The content/context argument of Section IV gives exactly that additivity, since it concerns the outcomes of any comparison, sharp or unsharp, and the law of symmetry gives each effect one weight in every measurement that contains it. The Born rule therefore holds in the $\mathbb{H}$ sector directly, by the same argument that gives it in the larger sectors.

Gleason's theorem returns the residual of Section II as a theorem of the Hilbert space. If a state fixed every finding, each projection would receive the weight 0 or 1. In dimension three or more, every weight that meets the two premises has the form $\mathrm{tr}(\rho P)$, which varies continuously as the projection turns. A continuous weight on the connected set of rays that takes only the values 0 and 1 is constant, and no complete measurement with three or more outcomes allows a constant weight. The result that, in dimension three or more, no weight taking only the values 0 and 1 meets Gleason's two premises is the Kochen-Specker theorem. Gleason's theorem gives the Kochen-Specker theorem in this form, as Bell noted (Bell, 1966; Kochen and Specker, 1967). The weights are single, by the law of symmetry; the values are not fixed, by the Incommensuration Theorem; and the theorem shows that the two statements belong together.

IV. Gleason's Premises from the Structure of Comparison

The third stage of the argument derives the two premises of Gleason's theorem from the structure of comparison: additivity from the six intrinsics, and again from repetition, and the single weight of a finding from the law of symmetry (the formal step, below).

The additivity condition states: if two outcomes of a measurement are mutually exclusive (corresponding to orthogonal subspaces), then the probability of obtaining one or the other is the sum of the individual probabilities.

This condition is so natural that it is often assumed without comment. But the additivity condition excludes, for instance, a weight for "outcome j or outcome k" that differs from the sum of the two weights. It does not exclude a weight that depends on which complete measurement contains the outcome; that exclusion is Gleason's other premise, which the formal step below supplies. The additivity condition is a specific structural requirement, and its derivation from a non-probabilistic starting point is the key step.

The argument proceeds as follows.

The six intrinsics (sameness and difference, content and context, subjective and objective) are necessary and sufficient for any comparison. A measurement is a specific comparison: the apparatus is the perspective from which the comparison is made (subjective), and the system is what is compared (objective); the property probed is the content, and the apparatus's setting and reference scale are the context; the result is the verdict of sameness or difference.

A measurement with n possible outcomes partitions the content space into n regions. Each outcome corresponds to one region. The partition is exhaustive (every possible content falls into exactly one region), because the content/context pair is a structural intrinsic: there is no content that escapes the partition and no region of the partition that fails to correspond to content.

Now consider two outcomes that are mutually exclusive: outcomes j and k with $j \neq k$. Their corresponding regions do not overlap (by the definition of mutual exclusivity). The compound event "outcome j or outcome k" corresponds to the union of the two regions.

The key step: the sameness/difference pair requires that the measure assigned to the union of two disjoint regions equals the sum of the measures assigned to each region individually.

The argument for this key step rests on the structural role of the sameness/difference pair. Sameness identifies what is common to two things being compared; difference identifies what distinguishes them. If two outcomes are different (mutually exclusive), then the measure of their disjunction must preserve their distinctness: it must be decomposable into the contributions of each. If the measure of the disjunction were *not* the sum of the parts, then there would be a structural interaction between the two outcomes that could not be attributed to either one alone. Such an interaction would violate the completeness of the content/context partition: the "interaction" between the two outcomes would be content that belongs to neither region, contradicting the exhaustiveness of the partition.

Therefore: the probability of the compound event is the sum of the probabilities of the individual events. The rule that the probabilities of exclusive events add is Gleason's additivity condition.

Additivity has a second route, from the identification of Section II alone. A probability is the stable relative frequency of an outcome over repetitions of one comparison. No repetition finds both of two exclusive outcomes, so in any run of repetitions the number that found one or the other is the number that found the first plus the number that found the second. Divided by the number of repetitions, and carried to the stable value, the weight of "j or k" is the weight of j plus the weight of k. For the same reason no weight is negative, and the weights of all the outcomes of one complete comparison sum to one. The argument from the intrinsics reaches additivity from content and context; this route reaches it from repetition; the two rest on different premises and give one result.

The argument concerns the outcomes of a comparison, not regions of the content space as such. An unsharp comparison still has exclusive and exhaustive outcomes: its record partitions into them, though the content does not. The measure of "outcome j or outcome k" is therefore the sum of the two measures, for effects as for projections; and by the law of symmetry (the formal step below) an effect has one weight in every measurement that contains it. Additivity on effects, with one weight for each effect in every measurement that contains the effect, is Busch's premise.

The formal step.

Gleason's theorem has two premises, and the argument supplies each. The first premise of Gleason's theorem, the non-contextuality of the weights, is that the weight of a finding depends only on the subspace that represents the finding. By the first premise, a projection P that belongs to two complete measurements, $\{P, Q_1, Q_2\}$ and $\{P, R_1, R_2\}$, receives one weight in both measurements. The non-contextuality of the weights is the law of symmetry applied to findings (Chapter 3, Section II). What is found as content is found as a sameness of content (Chapter 8, Section I). So the same content, found in two contexts, is the same finding. A difference of weight with no difference of content would be a difference that no comparison of the content could register. The first premise is stated here as an identification. The second premise is that within one complete measurement the weights of exclusive outcomes add, which is the argument above from the content/context partition and the sameness/difference pair, and again the count of repeated outcomes. With both premises the weights form a measure on the projections of each sector. The algebra has two non-commutative summands, from two sources: the continuously asymmetric mode requires a non-commutative summand, which $M_3(\mathbb{C})$ supplies, and the carrier condition requires the non-commutative division carrier, which is $\mathbb{H}$. On their sectors the observables are $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ (Chapter 6, Section VI), whose projections form non-Boolean lattices (lattices that fail the distributive law of classical logic), of dimension two and three. The theorem applies sector by sector: Gleason's in the three-dimensional sector, Busch's in the two-dimensional one, and the one-dimensional sector has a single outcome. In every sector the result is the Born rule, and on the full space of Chapter 6, by the same two premises, it is the Born rule for every finding.

The first premise has a second route, from the unitary transformations of the state space, which turn its directions and keep the size of every state. A turning that leaves the direction of an outcome where it is changes nothing about that outcome, so it cannot change how strongly

that outcome could be found. Two complete measurements that share a projection P, such as $\{P, Q_1, Q_2\}$ and $\{P, R_1, R_2\}$, differ by a turning of the other outcomes that leaves P where it is, so P receives one weight in both. Two further conditions come from what a comparison and a state are. The weights of the outcomes of one comparison, taken together, are the same under every turning, since a comparison finds exactly one outcome. An outcome in whose direction the state has no amount could not be found, so its weight is zero. In three or more dimensions, by Gleason's theorem (Section III), the one weight that meets these conditions for every state and every turning is the squared size of the state's amplitude in the direction of the outcome.

> **The Running Example: One Half and One Half**
>
> Prepare the silver atom of Chapter 3 with its spin up along x, and send it through a magnet turned along z. On $\mathbb{C}^2$ the weight of "up along z" is $|\langle \text{up along z} \mid \text{up along x}\rangle|^2 = \tfrac{1}{2}$, and so is the weight of "down along z"; Busch's theorem fixes these values on $\mathbb{C}^2$. The second route to additivity is visible on the screen. Of a thousand atoms, about five hundred land up and about five hundred land down; no atom lands in both places, so the count of "up or down" is the sum of the two counts, and it is all thousand.

V. The GNS Construction and the Born Rule

The GNS (Gel'fand-Naimark-Segal) construction, which provides the passage from the comparison algebra to the Hilbert space (Chapter 6), supplies the vector and the inner product in which the Born weight is written.

Recall the construction. Given the comparison algebra A_F and a state φ on A_F, the GNS theorem produces a Hilbert space H_φ, a representation π_φ, and a cyclic vector Ω_φ, such that:

$$\varphi(a) = \langle \Omega_\varphi, \pi_\varphi(a)\, \Omega_\varphi \rangle$$

for every element a in A_F.

The fundamental equation writes the value of a state on an element as an expectation in a vector. Applied to a projection P it gives $\varphi(P) = \|\pi_\varphi(P)\Omega_\varphi\|^2$, which is the Born weight once the value of the state on P is identified as the probability of the finding P. Sections II to IV supply that identification; the GNS construction supplies the vector and the inner product in which the weight is written. For a one-dimensional eigenspace spanned by a unit vector e_k, the weight is $|\langle e_k|\Omega_\varphi\rangle|^2$, the form of Postulate 6.

In the three-column dictionary (Chapter 6, Section I) the GNS construction is read in parallel with the Root Tautology and the Yoneda lemma: an algebra is faithfully represented by the totality of its states, as an object is determined by its morphisms, as interaction is comparison.

VI. The Three Layers of the Argument

The complete argument can now be assembled. It has three layers, each depending on the one before.

Layer 1: Necessity of probability. The ICT establishes that symmetry and continuity cannot both be perfected, so every lawful description leaves a residual: the state does not fix the finding. Gleason's theorem returns the same residual, in dimension three or more, as the Kochen-Specker theorem. On the internal labels the residual has an exact algebraic form: the state fixes which class of comparison is found and does not fix the finding within the class. Sameness applied to repetition gives each finding one stable relative frequency, its probability (an identification). Probability is structurally necessary.

Layer 2: Uniqueness of the Born rule. Gleason's theorem establishes that for Hilbert spaces of dimension three or greater, the Born rule is the *unique* probability measure that gives each closed subspace one weight and is additive on orthogonal subspaces. Gleason's theorem applies in the

three-dimensional sector, Busch's in the two-dimensional one, and Gleason's, with countable additivity, on the full infinite-dimensional space; the Born rule is the only option for every finding.

Layer 3: Derivation of Gleason's premises. The single weight of a finding follows from the law of symmetry, and additivity by two routes: from the content/context partition and the sameness/difference pair, and from the count of repeated outcomes. The content/context partition requires that mutually exclusive outcomes partition the content space exhaustively, and the sameness/difference pair requires that the measure decompose accordingly.

Together, these three layers derive the Born rule from the structure of comparison. No probabilistic framework is assumed at the start. Probability comes from the ICT and from sameness applied to repetition, and the specific probability measure is selected by the theorems of Gleason and Busch acting on the finite sectors and on the full Hilbert space that the GNS construction and the spacetime factor provide.

VII. Summary

The Born rule is not a free parameter of quantum mechanics. It is a structural consequence of the comparison relation, channeled through three results: the ICT, which with sameness applied to repetition forces probability; the theorems of Gleason and Busch, which select the Born rule uniquely, on every sector and on the full space; and the structure of comparison, which gives both premises of Gleason's theorem, the single weight of a finding from the law of symmetry and additivity from the six intrinsics and from the count of repeated outcomes. Gleason's theorem gives back the residual from which the argument began, as the Kochen-Specker theorem: in dimension three or more, no state fixes every finding.

The argument rests on two stated identifications: that the same comparison, repeated on the same state in the same context, gives each outcome one stable relative frequency; and that the same content found in two contexts is the same finding, with one weight. It runs from comparison to the Born rule without a probabilistic premise.

KEY CONCEPTS IN THIS CHAPTER

- **Born rule**: The prescription $p(k) = |\langle\psi|e_k\rangle|^2$ connecting quantum states to measurement probabilities. Postulate 6 of standard QM.
- **Gleason's theorem**: For Hilbert spaces of dimension ≥ 3, the Born rule is the unique probability measure that gives each closed subspace one weight and is additive on orthogonal subspaces (Gleason, 1957); it is applied here sector by sector.
- **Busch's theorem**: In every dimension, two included, the Born rule is the unique probability measure on effects that is additive over the outcomes of a measurement (Busch, 2003); it carries the result into the two-dimensional sector.
- **Non-contextuality of the weights**: The weight of a finding depends only on the subspace that represents it, whatever complete measurement contains it; Gleason's first premise, from the law of symmetry (an identification).
- **Probability from repetition**: The same comparison, performed on the same state in the same context, is the same comparison, so its outcomes recur with one stable relative frequency, the probability (an identification).
- **Additivity condition**: The requirement that probabilities of mutually exclusive outcomes sum to the probability of their disjunction. Gleason's second premise, from the content/context partition and the sameness/difference pair, and from the count of repeated outcomes.
- **Kochen-Specker theorem**: In dimension three or more, no assignment of the values 0 and 1 to every projection meets Gleason's two premises; the residual of the Incommensuration Theorem, returned as a theorem of the Hilbert space (Section III).
- **Content/context partition**: The structural division of every comparison into what it is about (content) and the respect in which the things compared are two (context). One of the three intrinsic pairs.

- **Three times**: Evolution time (transcendent), measurement time (immanent), geometric time (omniscient, time laid out as structure). Evolution and measurement are structurally distinct and cannot be reduced to each other. Evolution time is a parameter and not a content, so time is not an observable.

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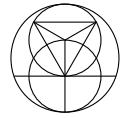
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Chapter 8

Superposition, Entanglement, and Spin

The Quantum Phenomena as Structural Consequences of the Comparison Algebra

ABSTRACT

The central quantum phenomena (superposition, entanglement, spin, interference, Bell nonlocality, and the statistics of identical particles) are standardly treated as empirical facts that the formalism accommodates but does not explain. This chapter shows that each phenomenon traces to a specific structural feature of the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ derived in the preceding chapters. Superposition follows from the vector-space structure of the GNS representation space, and the law of symmetry settles which superpositions are found. Entanglement follows from the tensor product of composite systems, which Chapter 6, Section VI derives. Spin and weak isospin both follow from the quaternions: the unit quaternions are $SU(2)$, acting on the spinor fields of spacetime as spin and on the $\mathbb{H}$ factor of the internal space as weak isospin. Interference follows from the linear structure of the representation space, and its phases from the complex type of the omniscient and transcendent carriers. Bell nonlocality follows from non-commuting observables on both sides, which both non-commutative carriers supply in the complex context of evaluation, together with the entangled states of composite systems. Fermi and Bose statistics follow from the principle of identity applied to the two poles of the partition, content interexclusionary and context intercompositional, and the real structure and the grading fix KO-dimension 6. In each case, the phenomenon is not postulated but derived: a structural consequence of the comparison relation and its algebraic encoding.

I. Superposition

Superposition is the most basic of quantum phenomena. A quantum system can exist in a state that is a complex linear combination of other states. The electron in a double-slit experiment passes through both slits simultaneously, or more precisely, its state is a superposition of the states corresponding to each slit separately.

In the Immanent Metaphysics, superposition is not a postulate. It is a consequence of the vector-space structure of the Hilbert space H_{F}, which is itself a consequence of the GNS construction applied to the comparison algebra.

The GNS construction takes the algebra A_{F} and a state φ and produces a Hilbert space H_{φ} whose elements are equivalence classes of algebra elements, with inner product $\langle a, b\rangle_{\varphi} = \varphi(a^{*}b)$. The Hilbert space inherits the linear structure of the algebra: if a and b are elements of A_{F}, then so is every real combination of them, and the GNS space, built on the complex span of the algebra, is closed under all complex combinations. The vector of a combination $\alpha a + \beta b$, for any complex numbers α and β, is the superposition of the vectors of a and b.

In IM terms, superposition expresses the fact that a comparison does not force a definite outcome until it is performed. Within each sector of A_{F}, a state can be any complex linear combination of that sector's states, and an outcome is actualized only when a specific comparison is made.

Superposition has a second route, from possibility itself. Possibility passes continuously from one direction of the state space to another (Chapter 7, Section II), so a state need not lie in one direction: it can lie between directions, as a combination of several directions. The coefficients of the combination are complex numbers, because the omniscient and transcendent carriers are conjugate and the identification of Chapter 5, Section III carries their conjugacy to the context of evaluation (Section V);

so each coefficient has a phase as well as a size. What could be found of a thing is therefore not just a choice among directions but an amount for each direction, each with its phase. Written in those directions, the set of amounts with their phases is the thing's state, and each amount with its phase is an amplitude.

Stated structurally, the comparison relation defines a space of possible contents (the Hilbert space H_{F}). A state is a point in this space. There is no axiom that restricts states to the basis vectors. The vector-space structure is inherited from the algebraic structure of comparison itself: specifically, from the fact that comparisons compose (algebraic multiplication) and scale (scalar multiplication by complex numbers).

Superposition is therefore not mysterious. It is the inevitable consequence of encoding comparison as a *-algebra. Restricting states to basis vectors would require a constraint beyond the superselection that the law of symmetry imposes (below), and the six intrinsics motivate none. Superposition is the default; definite outcomes are the exception, produced only when a specific comparison is performed.

Which superpositions are found

Within each sector any complex combination is a state; which states can be found is a further question, which the algebra alone does not settle. The law of symmetry settles it: what is found as content is found as a sameness of content across a difference of context. The law is carried to the algebra by two identifications. A content found is a ray of the space of states. A difference of context within one comparison is a symmetry that leaves the Dirac operator unchanged; for A_{F} these symmetries are electromagnetism and color.

A content found is therefore carried into itself by each such symmetry, up to a phase; the phase, taken as a function of the symmetry, is a character of its group, a map to the circle that respects composition. $\mathrm{SU}(3)$ has no non-trivial character, so a content found is a color singlet: only color singlets are found, and a color triplet is never superposed with a singlet.

A character of the circle is $\theta \mapsto e^{ik\theta}$, so a content found lies in one eigenspace of the charge: charge is superselected. The other symmetries of the quaternions do not leave the Dirac operator unchanged, so the law sets no condition through them: weak isospin is neither confined nor superselected (*Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 19, Section VIII).

II. Entanglement

Entanglement is the phenomenon in which two spatially separated systems share a joint quantum state that cannot be decomposed into independent states of the individual systems.

The mathematical expression of entanglement is straightforward. For two systems with comparison algebras A_1 and A_2, the composite system has algebra $A_1 \otimes A_2$ and Hilbert space $H_1 \otimes H_2$. A state in $H_1 \otimes H_2$ is *entangled* if it cannot be written as a tensor product $|\psi_1\rangle \otimes |\psi_2\rangle$. A mixed state of the composite is entangled when it is not a convex combination of product states.

In the content/context account of entanglement, when two systems are compared jointly (when a single comparison involves both systems as content) the joint comparison creates a structure that cannot be decomposed into independent comparisons of the parts.

That the joint comparison cannot be decomposed into independent comparisons of the parts is a structural consequence of the content/context partition applied to composite systems. If system 1 is measured (used as content) in a context that includes system 2, then the outcome of the measurement depends on the joint state of both systems. The content of the comparison is not localized in either system alone but distributed across the pair. The content/context partition is a partition of the *joint* system, not of the individual systems separately.

The tensor product structure of composite algebras is not an additional postulate. Chapter 6, Section VI derives it, once, for Postulate 4. The derivation identifies two systems with two comparisons that can be performed together without disturbing each other, and a theorem of operator algebra then makes the joint space the tensor product of the two modules and the joint algebra the tensor product of the two algebras. The GNS construction on a product state of the tensor product yields the tensor product of the GNS Hilbert spaces. Entangled states arise because the tensor product space $H_1 \otimes H_2$ is strictly larger than the set of product states $\{|\psi_1\rangle \otimes |\psi_2\rangle\}$.

In the language of the six intrinsics: entanglement is what occurs when the sameness/difference pair is applied to a composite system and the sameness between the parts exceeds what can be attributed to each part independently. The "excess sameness" is the entanglement. It is content that belongs to the joint comparison but to neither sub-comparison alone.

III. Bell Nonlocality

The Bell inequalities are constraints on correlations between measurement outcomes at spatially separated locations that any theory of local hidden variables must satisfy. Quantum mechanics violates them. The violation has been confirmed experimentally to extraordinary precision.

The structure of comparison provides a structural account of this violation. The comparison algebra A_F is non-commutative: there exist elements a and b in A_F such that $ab \neq ba$. The non-commutativity of A_F is not an accidental feature, and it has two sources.

The first source of the non-commutativity is the Incommensuration Theorem. In algebraic language: an observable of the continuously asymmetric mode commutes with the Dirac operator without being central, and an algebra has such observables only if it is non-commutative (Chapter 5, Section II); the carrier $M_3(\mathbb{C})$ supplies them. The second source of the non-commutativity is the carrier condition, which requires the non-commutative division carrier $\mathbb{H}$ (Chapter 5, Section V). The ob-

servables of the qubit, the Pauli matrices, belong to the complex algebra that $\mathbb{H}$ generates on its module in the complex context of evaluation (Chapter 6, Section VI). So the qubit's non-commuting observables rest on the carrier condition and the complex context, and its entangled states on joint comparison.

Bell's theorem assumes that the outcome on one side does not depend on the setting chosen on the other, and that outcomes are fixed in advance by variables the two sides share (Bell, 1964). In the form of Clauser, Horne, Shimony and Holt (1969), the CHSH inequality, each of two parties chooses one of two measurements with outcomes ± 1, and under those assumptions a sum of four correlations cannot exceed 2. In $A_{\mathrm{F}} \otimes A_{\mathrm{F}}$ a violation needs two things, and the algebra supplies both: on each side, a pair of ± 1 observables that do not commute (on the module of $\mathbb{H}$, any two Pauli directions at an angle), and a joint state that is not a product, which the tensor product admits (Section II).

An identity of Landau (1987) shows why both are needed. Let A and A' be observables of one party and B and B' observables of the other, each of square one, with the observables of one party commuting with those of the other. The CHSH operator $C = A(B + B') + A'(B - B')$ satisfies $C^2 = 4 - [A, A'][B, B']$. If either commutator is zero, $C^2 = 4$, and no state gives a value above 2; and a product state gives no value above 2, whatever the observables. The violation of Bell inequalities is therefore a structural consequence of two things: non-commuting observables on both sides, which both non-commutative carriers supply in the complex context of evaluation; and the entangled states of the tensor product.

The foundations read the same result at the level of character. Bell's theorem shows that lawfulness, a symmetry of physical law, and locality, a continuity of causal connection, cannot both hold in quantum mechanics (*Incommensuration*, Chapter 15, Section III). Physics keeps its universal laws and holds the correlations within the limit of causal signaling: each side's outcomes, taken alone, are random, the correlation appears only when the two records are brought together by signals that keep to the lim-

it, and "nothing that carries a finding outruns the limit" (*Measurement*, Chapter 5, Section IX). No global order of the two distant measurements is needed. Each comparison has its own measurement time (Chapter 7, Section II), and the two records join only where the signals that carry them meet.

The correlation is not located at either place. The two systems share one possibility, and possibility is bound to the stance from which a comparison is made, not to a place (*Measurement*, Chapter 4), so nothing passes between the places to make the correlation. No finding is sent through the shared possibility. A unitary transformation made on the system at one place acts on observables that commute with every observable of the other place, so it leaves the weights of the outcomes at the other place unchanged, and no comparison made at the other place can register what was done. The correlation is in what could be found, and it shows only when the findings at both places are compared.

On the Tsirelson bound.

Quantum mechanics does not violate the Bell inequalities by an arbitrary amount. The maximum violation is bounded by the Tsirelson bound ($2\sqrt{2}$ for the CHSH inequality, compared to the classical bound of 2). On the Hilbert space of Chapter 6 the bound is Cirel'son's theorem (1980): when each of two parties measures one of two observables with outcomes ± 1, and the observables of one party commute with those of the other, the CHSH value cannot exceed $2\sqrt{2}$. Landau's identity, $C^2 = 4 - [A, A'][B, B']$, gives the bound $2\sqrt{2}$ in two steps. The norm of an operator is the largest factor by which it can multiply the length of a state vector; the norm of an observable is the largest absolute value of its outcomes. An observable with outcomes ± 1 has square one, and its norm is one. The first step bounds each commutator. The norm of a difference is at most the sum of the norms, and the norm of a product is at most the product of the norms. So the commutator $[A, A'] = AA' - A'A$ has norm at most $1 + 1 = 2$, and the commutator $[B, B']$ has norm at most 2 in the same way. The second step bounds C. By the identity, C^2 has norm at most $4 + 2 \times 2 = 8$. The operator C is self-adjoint, and the norm of a self-adjoint operator is the square root of the norm of its square, so the norm of C is at most $\sqrt{8} = 2\sqrt{2}$. In every state, the expected value of C lies between minus its norm and its norm, so no state gives a CHSH value whose absolute value exceeds $2\sqrt{2}$.

The algebra says where the bound can be reached. A party whose two observables commute, as every observable of the commutative carrier $\mathbb{C}$ does, gives back the classical bound 2. The value exceeds 2 only when each party's two observables fail to commute and the joint state is entangled, which the non-commutative carriers $\mathbb{H}$ and $\mathrm{M}_3(\mathbb{C})$ allow. On $A_\mathrm{F} \otimes A_\mathrm{F}$ the classical bound 2 holds in every sector in which $\mathbb{C}$ appears on either side, and $2\sqrt{2}$ is reached in every sector in which both sides carry a non-commutative carrier. The smallest such case is the module of $\mathbb{H}$, the qubit, where the singlet state, with spin observables at suitable angles, reaches $2\sqrt{2}$.

For the suitable angles, let the first party measure spin along the z-axis (A) or along the x-axis (A'). Let the second party measure spin along the direction halfway between the z-axis and the x-axis (B), or along the direction halfway between the z-axis and the reverse of the x-axis (B'). Both directions lie in the plane of the two axes, at 45 degrees from the z-axis. The singlet state is $(|01\rangle - |10\rangle)/\sqrt{2}$, where $|0\rangle$ and $|1\rangle$ are the states of one spin up and down along the z-axis; in the singlet state the two spins are found opposite along every direction. When the two parties measure spin along two unit directions in the singlet state, the correlation of their outcomes is minus the cosine of the angle between the directions. A with B, A with B' and A' with B are each at 45 degrees, so each of those correlations is $-1/\sqrt{2}$; A' with B' is at 135 degrees, so its correlation is $+1/\sqrt{2}$. The CHSH value, the expected value of C, adds the first three correlations and subtracts the fourth: $-3/\sqrt{2} - 1/\sqrt{2} = -4/\sqrt{2} = -2\sqrt{2}$, whose absolute value is $2\sqrt{2}$. Each party's two directions are perpendicular, and perpendicular directions make each commutator as large as the first step allows. For spin observables along two unit directions, the commutator is 2i times the spin observable along the cross product of the directions. Its norm is two times the sine of the angle between the directions, and the norm is two exactly when the directions are perpendicular. With both commutators of norm two, C^2 takes the value 8 in the singlet state, the largest value the second step allows.

The carrier $M_3(\mathbb{C})$ reaches the same value, as a value of the CHSH operator in a state of $A_F \otimes A_F$. Among its self-adjoint elements of square one are those that act on the plane of the first two basis directions of $\mathbb{C}^3$ as the spin observables above act on $\mathbb{C}^2$, and act on the third basis direction as $+1$. So either non-commutative carrier supplies the observables of the computation above: $\mathbb{H}$, through the complex algebra it generates on its module $\mathbb{C}^2$, or $M_3(\mathbb{C})$, acting on that plane of $\mathbb{C}^3$. The state of $A_F \otimes A_F$ of the same form as the singlet, built with the first two basis directions of $\mathbb{C}^3$ in place of $|0\rangle$ and $|1\rangle$ on each side that uses $M_3(\mathbb{C})$, gives the same four correlations and the same CHSH value, $-2\sqrt{2}$. The value $2\sqrt{2}$ is therefore

reached, in a state of $A_F \otimes A_{F'}$ in each of the four sectors in which both sides carry a non-commutative carrier: $\mathbb{H}$ with $\mathbb{H}$, $\mathbb{H}$ with $M_3(\mathbb{C})$, $M_3(\mathbb{C})$ with $\mathbb{H}$, and $M_3(\mathbb{C})$ with $M_3(\mathbb{C})$.

IV. Spin

Spin is the intrinsic angular momentum of a particle, and it has no classical analogue. In quantum field theory the spin-statistics theorem ties half-integer spin to Fermi statistics; Section VI reaches the same assignment from content and context, without that theorem. An electron has spin one half: its spin angular momentum takes the values $+\hbar/2$ and $-\hbar/2$ along any axis of measurement.

In the ICT framework, spin comes from the quaternions. The unit quaternions form the Lie group $SU(2)$. Every unit quaternion $q = a + bi + cj + dk$ with $a^2 + b^2 + c^2 + d^2 = 1$ acts on $\mathbb{C}^2$ as a unitary matrix:

$$q \mapsto \begin{bmatrix} a + bi, & c + di \\ -c + di, & a - bi \end{bmatrix}$$

In this representation the quaternionic units i, j, k are the matrices $i\sigma_z$, $i\sigma_y$, $i\sigma_x$, where σ_x, σ_y, σ_z are the Pauli matrices. Two groups $SU(2)$ act in physics, and both are the unit quaternions. The first of the two groups $SU(2)$ is the group of spin. The group of spin acts on the spinor fields of spacetime, the space $L^2(M, S)$ of Chapter 6, as the double cover of the rotations of space. The second of the two groups $SU(2)$ is the group of weak isospin. The group of weak isospin acts on the $\mathbb{H}$ factor of A_F, on the internal space $\mathbb{C}^2$ of Chapter 6. The two actions are distinct in physics. A right-handed electron is a weak-isospin singlet and still has spin one half; a quark has spin one half whatever its color.

What the IM explains is why the two are the same group. In the focus of the two laws, the immanent modality is carried by the quaternions. The ICT requires both of its modes, the continuous asymmetric mode of spacetime and the discontinuously symmetric mode of the internal space,

and the quaternionic structure of the immanent appears once in each: as the rotations of spinors in the first, and as weak isospin in the second. The rotations belong to the arena, which supplies the frames of directions; the spin itself is a label of the content, kept across every change of context and found in steps, as *Measurement*, Chapter 9, Section V, states: "Spin is content." In *The Gravitational Passage* the three directions of space come from the three transitions of the cycle of Axiom II, and the one direction of time from the direction of the cycle (Chapter 10).

Spin is not introduced as an additional degree of freedom appended to the quantum formalism. In the focus of the two laws, it is an intrinsic feature of the immanent modality, the modality of process, of relation, of the actual. The immanent modality mediates between extremes, and its quaternionic character gives that mediation a handedness: the order of composition matters, since quaternion multiplication is non-commutative. Spin and weak isospin are, in this precise sense, the modality of relation made visible, once in each mode: in spacetime as the angular momentum of spin, and in the internal space as weak isospin. The chain from the immanent modality to the existence of spin is direct and contains no free parameters.

Which modality spin belongs to is set by the frame in which the carriers are labeled. In the frame of measurement of *The Story of the Particles*, a rotation of the same labeling (Chapter 3, Section V), $\mathbb{H}$ carries the transcendent, and spin is the transcendent label of a particle; its unit, $\hbar$, is the microscopic limit of the domain (*The Kinds of Particle*, Chapter 4, Section III). The focus of the two laws and the frame of measurement give two bindings of one structure. The step from the quaternions to spin one half uses the unit group of $\mathbb{H}$, $\mathrm{SU}(2)$, and not the modality that $\mathbb{H}$ carries, so it holds in both.

A second route reaches spin one half from premises that never mention the quaternions. *Measurement*, Chapter 9, Section V, takes the three directions of space, which give every turning one axis, and the one ordered direction of time, which gives it one of two senses, and identifies

the least step of intrinsic circulation with ħ. The values found about any axis then form a ladder, symmetric about zero, with steps of one unit. A ladder of that kind either has a rung at zero, and its least non-zero value is one, or has none, and then its least magnitude is one half, found with the two signs $+½$ and $-½$; the smallest ladder of that kind has exactly these two outcomes. The two routes are independent, and they reach the same least value and the same two outcomes.

The Bloch sphere as comparison space.

The ray space of $\mathbb{H}$, the sphere of complex lines in $\mathbb{C}^2$, is the Bloch sphere S^2. Every point on S^2 is a ray of the module of $\mathbb{H}$; in the focus of the two laws it is a maximal comparison along the immanent modality: a direction along which the immanent modality is oriented. For spin, the points are the directions along which spin can be measured; for isospin, they are the orientations of the internal doublet. The north and south poles are the two eigenstates of any chosen measurement axis, and the equator holds the balanced superpositions of the two eigenstates. The group $SU(2)$ acts on S^2, through the rotations $SO(3)$, as the group of all rotations of the comparison axis. Spin one half is the fundamental representation of $SU(2)$ on $\mathbb{C}^2$, the representation that the GNS construction on $\mathbb{H}$ produces in Chapter 6. Its defining property, that a rotation by 360° returns the state vector to its negative and a rotation by 720° returns it to itself, is a consequence of $SU(2)$ being the double cover of $SO(3)$. The Bloch sphere is the ray space of the immanent carrier, and its geometry ($S^2 = SU(2)/U(1)$) is derived from the algebraic structure of the comparison relation.

Spin as found, and the double cover tested.

Spin is known through what comparison finds. In the experiment of Gerlach and Stern (1922), silver atoms sent through an inhomogeneous magnetic field reach the screen in two separate traces, not in the continuous spread that a magnetic moment free to point in every direction would give. A comparison of spin along one axis has two outcomes, discrete and exact: that is the discontinuously symmetric character that Chapter 4, Section III gives to what is found, while the states between the two outcomes, the points of the Bloch sphere, belong to the continuous formalism. The double cover is tested directly by neutron interferometry. A beam of neutrons is split in two, the spins in one path are rotated about an axis by a magnetic field, and the two paths are recombined. A rotation by 360° shifts the interference pattern by half a period, because the state vector of the rotated path changes sign, and a rotation by 720° restores the pattern, as the double cover requires (Rauch and colleagues, 1975; Werner and colleagues, 1975).

V. Interference

Quantum interference is the phenomenon by which probability amplitudes add before the probabilities are computed, producing constructive and destructive patterns.

The phenomenon traces to the linear structure of H_F, and its phases to the complex structure of the inner product.

The GNS inner product is: $\langle a, b \rangle_\varphi = \varphi(a^* b)$. The state φ is a positive linear functional on A_F, which means that $\varphi(a^* a) \geq 0$ for all a. The inner product takes values in $\mathbb{C}$, the complex numbers, not in $\mathbb{R}$.

Interference comes from the linear structure of H_F: amplitudes add before probabilities are taken. When a system can reach a final state through two distinct paths, the total amplitude is the sum of the path amplitudes: $A = A_1 + A_2$. The probability is $|A|^2 = |A_1|^2 + |A_2|^2 + 2\,\mathrm{Re}(A_1^* A_2)$. The last

term, the interference term, can be positive (constructive) or negative (destructive), with real amplitudes as with complex ones. The complex field gives every amplitude a phase on a circle rather than a sign, and it is that circle of phases, not interference as such, that distinguishes complex from real quantum theory in experiment (Chapter 1).

In the IM framework, the complex-valuedness of the inner product is not a choice, and it has two sources, each named. The Hilbert space is complex because the context of evaluation is a faithful complex representation, which Chapter 5, Section III states as an identification: a conjugate pair carried by the context needs a conjugation on it, a map that is antilinear and not linear, and a space has one only if its scalars are not real. Within that space, the conjugacy clause of Axiom I, read in Chapter 5 as a condition on the algebra, makes the omniscient and transcendent carriers of complex type: the omniscient carrier is $\mathbb{C}$, not $\mathbb{R}$, and the transcendent carrier is $M_3(\mathbb{C})$. Each comes with a conjugate distinct from it. The question "why complex numbers and not real numbers?" is answered: because the omniscient and the transcendent are conjugate, and the identification of Chapter 5, Section III carries their conjugacy to the context of evaluation. Network experiments confirm that nature's quantum theory is complex and not real (Chapter 1).

Interference is therefore a structural consequence of the linear structure of the representation space, which every Hilbert space built by the GNS construction has. The phases of interference are complex because the context of evaluation is complex, and the comparison algebra is represented on it.

Interference is found in every two-path experiment: with single electrons, whose dots build up the pattern one by one (Tonomura, Endo, Matsuda, Kawasaki and Ezawa, 1989); with atoms (Carnal and Mlynek, 1991); and with molecules, from the sixty carbon atoms of C_{60} (Arndt and colleagues, 1999) to molecules of more than 25,000 atomic mass units (Fein and colleagues, 2019).

VI. Identical Particles and Statistics

Identical particles, particles that are in principle indistinguishable, obey either Bose-Einstein statistics or Fermi-Dirac statistics. Bosons can occupy the same quantum state. Fermions cannot: no two fermions can share the same quantum numbers. In quantum field theory the spin-statistics theorem ties the two kinds to spin: integer spin with Bose statistics, half-integer spin with Fermi statistics.

The existence of identical particles follows from the structure of the comparison algebra. A particle is an excitation of the Hilbert space $L^2(M, S) \otimes \mathscr{H}_F$: a state created by the action of a field operator on the vacuum. Two excitations in the same sector carry the same quantum numbers, the same gauge charges, the same spin, and nothing in the algebra distinguishes them. What no comparison can distinguish is one kind of thing.

The statistics from content and context

The two statistics come from the principle of identity, applied to the two poles of the content/context partition, as *The Story of the Particles* applies it. The principle of identity says that what no comparison can distinguish, in any way at all, even in principle, is one thing and not two; *Measurement*, Chapter 11, Section IV, states it for differences: "a difference that no comparison can register is no difference." The two poles differ in how their instances stand to one another: "Content is local and interexclusionary, and understood in terms of symmetry: two contents exclude one another from the same place. Context is non-local and intercompositional, and understood in terms of continuity: contexts compose with one another" (*Incommensuration*, P7; *An Immanent Metaphysics*, aphorism 1.46-12).

Applied to the content pole, whose instances are separated from one another, the principle gives exclusion: two instances of content of one kind in one state could not be told apart, so they would be one instance, and the orthogonal projection onto the many-instance states annihilates every

tensor with two equal factors. Applied to the context pole, whose instances are not separated, the principle gives addition: context contributed twice in one state is one context of twice the amount, so the many-instance states contain every tensor $\psi \otimes \ldots \otimes \psi$, and are the smallest space that does. The two clauses are stated as one identification on the space of many-instance states.

A theorem of linear algebra finishes the step. The exchange of two factors of $V \otimes V$ has eigenvalues $+1$ and -1, with the symmetric tensors and the antisymmetric tensors as its eigenspaces, and the tensors $\psi \otimes \psi$ span exactly the symmetric ones; so the largest subspace whose orthogonal projection annihilates every $\psi \otimes \psi$ is the space of antisymmetric tensors, and the smallest subspace containing every $\psi \otimes \psi$ is the space of symmetric tensors. So the clause for content leaves exactly the antisymmetric tensors, and the clause for context requires exactly the symmetric ones. Exchanging two instances of content multiplies the state by -1; exchanging two instances of context leaves it fixed. The two exchange rules are Fermi statistics, for content, and Bose statistics, for context (*Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 19, Section III).

Which fields stand on which pole is a step of its own, and it is made by position in the act of comparison, as *The Story of the Particles* makes it. A comparison has three positions: the act of comparing, the thing compared, and the thing compared against. The fermion fields are the states of what is compared and of what it is compared against: the particles of matter, on the content pole. The gauge fields and the Higgs field are the fluctuations of the Dirac operator, the operator that measures difference between states, and so they carry the act of comparing; the act belongs with the framework within which comparison takes place, which is context, so these fields stand on the context pole. Placing the gauge fields and the Higgs field on the context pole is an identification. I make that identification in the frame of the positions of comparison. It is a different frame from the frame of the six measures, in which light is content, a carrier of force, and the arena of space and time is the context (*Measurement*, Chapter 3, Section II). *Measurement* holds both readings, each in its frame:

in the frame of the six measures a photon is content, and "in the frame of the smallest physical comparison it is the interaction that joins emitter and receiver" (Chapter 3, Section IX). The statistics belong to the frame of the positions, and every claim of this section about the fields is made in it. The observation that bears on the identification: electrons exclude one another from a state, as the periodic table shows, and photons add in one state, as the light of a laser shows.

The spins are a separate fact: the fermion fields are spinors, of spin one half; the gauge fields are one-forms, of spin one; and the Higgs field is a function, of spin zero. So matter has spin one half and Fermi statistics, and the fields of interaction have integer spin and Bose statistics, the assignment the spin-statistics theorem reaches from relativistic quantum field theory. Here it comes from the principle of identity applied to the two poles of content and context, and the theorem of Pauli is not used.

A second route reaches the same statistics from the turn of a spinor. Exchanging two identical objects, each carrying its connection to its surroundings, is the same, for their placements, as turning one of them through a full turn (Finkelstein and Rubinstein, 1968). A rotation by 360° reverses the sign of a spinor of spin one half (Section IV), so exchanging two contents of one kind reverses the state of the pair. If the two were in the same state, the exchanged pair would be the same pair, since nothing tells them apart, and yet reversed; a state equal to its own reversal has amplitude zero and cannot be found, so two contents of one kind cannot share a state. The gauge fields and the Higgs field return to themselves after one full turn, so exchange restores them, and any number of them can share one state. The two routes rest on different premises, the principle of identity at the two poles and the turn of a spinor, and they give one result.

The real structure and KO-dimension 6.

Two identifications carry the pairs to the triple: the real structure J carries the two terms of a state, and the grading γ carries content and context. The real structure J of the finite triple is an antilinear operator on its Hilbert space; it exchanges the two terms of a state, toward the thing compared against and toward the thing compared, and on states it is charge conjugation. The grading γ is a self-adjoint operator of square one that divides the Hilbert space of the triple into two halves. Three sign conditions follow from what they carry. $J^2 = +1$: exchanging the two terms of a state twice returns the state. $J\gamma = -\gamma J$: exchanging the two terms exchanges content and context. $JD = DJ$: the act of comparison carries no order of the two terms, and so is indifferent to their exchange. These three signs hold for a real even triple exactly when its KO-dimension is 6 modulo 8 (*Non-Commutative Geometry and the Immanent Metaphysics*, Proposition 4.2); the KO-dimension of a real spectral triple is the class modulo 8 that its signs of this kind determine. KO-dimension 6 is also the value that Connes and Barrett found for the Standard Model from the removal of fermion doubling (Connes, 2006; Barrett, 2007); the two reasons are independent, and they select the same signs. Chapter 6, Section VII reads the same exchange as the conjugacy of Axiom I carried to the states, from which the antiparticles follow.

VII. The Unity of Phenomena

The six phenomena discussed in this chapter (superposition, entanglement, Bell nonlocality, spin, interference, and identical-particle statistics) are standardly treated as separate features of the quantum formalism, each requiring its own explanation or interpretive gloss.

In the IM framework, they are not separate features. They are aspects of a single structure: the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, its GNS representation, the Hilbert space that the representation produces, and the spectral triple built on it.

In the map from the comparison algebra to the phenomena, each phenomenon has an algebraic source and an origin in the structure of comparison:

Phenomenon	**Algebraic source**	**IM structural origin**
Superposition	Vector-space structure of each sector of H_F	Linearity of the representation; the law of symmetry for what is found
Entanglement	Tensor product $A_1 \otimes A_2$	Joint comparison; the content/context partition of the joint system
Bell nonlocality	Non-commuting observables on both sides, from the two non-commutative carriers, with an entangled state	Incommensuration Theorem (ICT) and the carrier condition (Axiom III)
Spin	Unit quaternions: SU(2) on spinors and on $\mathbb{H}$	Immanent modality (in the focus of the two laws), present in both modes
Interference	Linear structure of H_F; complex inner product	Phases from the complex type of the omniscient and transcendent carriers (Axiom I)
Statistics	Exterior and symmetric powers	The principle of identity on each pole: content interexclusionary, context intercompositional; the fields placed on the poles by position in the act of comparison (identifications)

The table shows that no phenomenon requires a postulate of quantum mechanics. Each traces to the comparison algebra, its representation and the spectral triple built on it, together with the premises of the Immanent Metaphysics that the chapter names: the law of symmetry, the principle of identity, and the interexclusion of content and intercomposition of con-

text. The algebra is itself derived from the structure of comparison, through the Incommensuration Theorem, the three axioms and the three modalities, through the identifications of Chapter 5.

VIII. Summary

The quantum phenomena are not mysterious additions to the formalism. They are structural consequences of the comparison algebra.

Superposition is what happens when a comparison does not force a definite outcome, and the law of symmetry settles which superpositions are found. Entanglement is what happens when a joint comparison exceeds the sum of its parts. In the focus of the two laws, spin and weak isospin are the modality of relation made algebraically visible, once in each mode. In the frame of measurement of *The Story of the Particles*, spin is the transcendent label of a particle. Interference is the addition of amplitudes; its phases are complex. Bell nonlocality is the experimental signature of non-commutativity on both sides of an entangled state. Identical-particle statistics are the principle of identity applied to the two poles of content and context.

None of these phenomena requires a postulate of quantum mechanics. Each traces to the comparison algebra, its representation and the spectral triple built on it, together with the law of symmetry, the principle of identity, and the interexclusion of content and intercomposition of context. The algebra was derived from the structure of comparison, through the Incommensuration Theorem, the three axioms and the identifications of Chapter 5. The phenomena follow.

KEY CONCEPTS IN THIS CHAPTER

- **Superposition**: The ability of a quantum state to be a complex linear combination of other states. Derived from the vector-space structure of the Hilbert space; only color singlets are found, and electric charge is superselected, by the law of symmetry.
- **Entanglement**: A joint state of two systems that cannot be decomposed into independent states of the parts. Derived from the tensor product, which Chapter 6, Section VI derives from the composition of two comparisons that do not disturb each other.
- **Bell nonlocality**: The violation of Bell inequalities by entangled quantum systems. Derived from non-commuting observables on both sides, which both non-commutative carriers supply in the complex context of evaluation, together with an entangled state.
- **Spin**: The intrinsic angular momentum of quantum particles. Derived from the quaternions: unit quaternions = $\mathrm{SU}(2)$, acting on the spinor fields of spacetime; the same group acts on the $\mathbb{H}$ factor as weak isospin. Spin one half is the fundamental representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$; a rotation by 360° reverses the sign of the state vector, as neutron interferometry confirms.
- **Interference**: The constructive and destructive combination of probability amplitudes. Derived from the linear structure of the Hilbert space: amplitudes add before probabilities are taken. Its phases are complex because the context of evaluation is complex (Chapter 5, Section III).
- **Identical particles**: Particles that are in principle indistinguishable. A consequence of the finite sectoral structure of A_{F}: excitations in the same sector carry identical quantum numbers.
- **Statistics**: Fermi statistics for content, Bose statistics for context, from the principle of identity applied to the two poles, content interexclusionary and context intercompositional, with the fields placed on the poles by position in the act of comparison; Pauli's theorem is not used.

- **KO-dimension 6**: The signs of the real structure J and the grading γ, fixed by the two terms of a state and by content and context.
- **Non-commutativity**: The property that $ab \neq ba$ for certain elements of A_F. It has two sources: the Incommensuration Theorem, since an observable of the continuously asymmetric mode commutes with the Dirac operator without being central, which only a non-commutative algebra allows ($M_3(\mathbb{C})$); and the carrier condition, which requires $\mathbb{H}$. On both sides of an entangled state, it is the condition of Bell violation.
- **Tsirelson bound**: The maximum quantum violation of the CHSH Bell inequality ($2\sqrt{2}$). Cirel'son's theorem on the Hilbert space of Chapter 6; it is reached only where both parties have non-commuting observables, the smallest case being the qubit. It follows from Landau's identity, because the commutator of two observables with outcomes ± 1 has norm at most two; and, in a state of $A_F \otimes A_F$, it is reached with either non-commutative carrier, $\mathbb{H}$ or $M_3(\mathbb{C})$, on each side.

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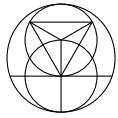
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Chapter 9

The Measurement Problem Dissolved

Why the Problem Was Never a Problem of Physics but a Problem of Starting Point

ABSTRACT

The measurement problem (the apparent inconsistency between the continuous unitary evolution of quantum states and the discontinuous collapse upon measurement) has generated a century of interpretive debate. Every major interpretation of quantum mechanics (Copenhagen, many-worlds, Bohmian mechanics, QBism, consistent histories) is an attempt to resolve this inconsistency. This chapter argues that the measurement problem is not a problem of physics but a problem of starting point. It arises because the standard formulation assembles its postulates from different sources (Hilbert's theory of integral equations for the state space, the spectral theory of operators for the observables, Hamiltonian mechanics for the dynamics, Born's probabilistic interpretation for the Born rule, and von Neumann's projection postulate, drawn from laboratory practice, for the collapse rule) and these sources are not structurally compatible. When quantum mechanics is derived from the comparison relation, as in the preceding chapters, the contradiction among the postulates does not arise, and what remains is the residual that the Incommensuration Theorem guarantees, which is no contradiction. The pair of the subjective and the objective, two of the six intrinsics, provides the structural distinction between the measured system and the measuring apparatus from the outset. The directed cycle of Axiom II makes the measurement transition structurally irreversible without invoking a separate dynamical law. The measurement problem dissolves not because this chapter solves it but because the derivation removes the conditions under which it was formulated. I take the case of Wigner's friend, derive the general form of a measurement's effect on a state, and place six interpretations on a map of the modalities each suppresses.

I. What the Measurement Problem Actually Is

Two things go by the name. The first thing that goes by the name of the measurement problem is a contradiction among the postulates. Postulate 3 says that every closed evolution is unitary, and Postulates 5 and 6 say that a measurement is not unitary. The second thing that goes by the name of the measurement problem is the gap between a continuous formalism and a definite finding. In *Measurement* I call that gap the residual. I dissolve the first, and state the second as what it is: no contradiction, and the place where the Born rule stands.

The measurement problem is often described informally as the question "why does the quantum state collapse?" But this informal description obscures the precise logical structure of the problem.

The problem is not that collapse is mysterious. The problem is that the postulates of quantum mechanics are internally inconsistent.

Postulate 3 states that the time evolution of a closed quantum system is governed by a unitary operator: $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$. Unitary evolution is linear, deterministic, and reversible. A superposition remains a superposition under unitary evolution.

Postulates 5 and 6 state that when a measurement is performed, the state "collapses" to one of the eigenstates of the observable, and the probability of each outcome is given by the Born rule. Collapse is nonlinear, probabilistic, and irreversible. A superposition is destroyed.

The problem: a measuring apparatus is a physical system. It is made of atoms, which are governed by quantum mechanics. The interaction between the apparatus and the measured system is a physical interaction, which should be governed by Postulate 3 (unitary evolution). But Postulate 3 preserves superposition, while Postulates 5 and 6 destroy it.

If the measurement interaction is unitary, then the composite system (apparatus + measured system) should evolve into a superposition of all possible outcomes. The apparatus should be in a superposition of having registered "spin up" and "spin down" simultaneously. Instead, we observe a single definite outcome.

The measurement problem is the demand for a consistent account of how definite outcomes arise from a formalism that preserves superposition.

> **The logical structure of the inconsistency.**
> The inconsistency can be stated as a trilemma (Maudlin, 1995). Three propositions cannot all be true: (1) the quantum state gives a complete description of the system; (2) the quantum state always evolves unitarily; (3) measurements always have definite outcomes. Every interpretation of quantum mechanics denies one of the three: Copenhagen denies (2); many-worlds denies (3); hidden-variable theories deny (1). In the derived formulation each proposition is said of one modality. (1) The state is the complete laid-out record of what has been found and of what a setting makes possible. (2) Between comparisons, in evolution time (transcendent), the state evolves unitarily. (3) Each comparison, in measurement time (immanent), has one definite outcome. The three conflict only when they are said of one modality; Axiom III keeps the modalities distinct, so all three hold.

II. The Source of the Problem

The measurement problem exists because the six postulates of quantum mechanics were not derived from a common source.

Historically (Chapter 1, Section III), the Hilbert-space postulate came from Hilbert's theory of integral equations (1906), axiomatized by von Neumann (1932). The self-adjoint operator postulate came from von Neumann's spectral theorem. The unitary evolution postulate is Schrödinger's

equation (1926), modeled on Hamilton's classical mechanics. The tensor product postulate is the composition rule made explicit by von Neumann (1932) and Schrödinger (1935). The Born rule (1926) was a guess: brilliantly confirmed by experiment, but still a guess. The collapse postulate is von Neumann's projection postulate (1932), a rule drawn from laboratory practice, building on Heisenberg (1927) and Dirac (1930), with Lüders' rule (1951) for the degenerate case.

Each postulate has a different origin. Each encodes a different intuition. The postulates were assembled into a formalism that works extraordinarily well for predicting experimental outcomes, but they were never shown to be mutually consistent as a logical system.

The measurement problem is the manifestation of this inconsistency. Postulate 3 comes from the classical theory of dynamics. Postulates 5 and 6 come from the interpretation of laboratory findings. The two sources encode different assumptions about what "measurement" is and how it relates to dynamics. When these assumptions collide, the measurement problem appears.

The standard interpretations attempt to resolve the collision by modifying or reinterpreting one or more postulates. The IM approach is different. It derives the formalism from a single source, the structure of comparison, and the derived formalism does not contain the inconsistency.

III. The Subjective and the Objective

The key structural resource for dissolving the measurement problem is the pair of the subjective and the objective, one of the three intrinsic pairs of the comparison relation.

Every comparison has a side from which it is made and a side about which it is made. The apparatus is the subjective side, the perspective from which the comparison is made; the system is the objective side, what is compared. The property of the system that is probed is the content, and

the setting of the apparatus, its places, times and orientations, is the context (Chapter 7, Section IV; *Comparison; The Necessary Basis*, Chapter 4, Section VII).

In standard quantum mechanics, the system/apparatus distinction is introduced pragmatically and is never given a structural foundation. The "Heisenberg cut" (the boundary between the quantum system and the classical apparatus) is placed by the experimenter at their convenience. It can be moved: the apparatus can be included in the quantum description, but then a new apparatus is needed to observe the old one, and the cut is only pushed further out. There is no principled location for the cut, because the formalism does not structurally distinguish system from apparatus.

In the IM framework, the division between the side that compares and the side compared is not introduced pragmatically. It is a structural intrinsic of the comparison relation. Every comparison, every measurement, has a subjective side and an objective side by structural necessity, since the two are inseparable. The division is not a choice made by the experimenter. It is a feature of the comparison itself.

Where the division falls is set by the comparison performed. The subjective and the objective are found by perspective (*Comparison; The Necessary Basis*, Chapter 4, Section VI): an instrument that is the subjective side of one comparison is the objective side of another, as it is when a second instrument reads the first. So the division is not fixed or unique for a given physical situation, and *some* division must exist for any comparison to occur.

The pair of the subjective and the objective dissolves the Heisenberg cut problem. Every comparison has a cut, because every comparison has a subjective and an objective side; the comparison performed sets where it lies. Moving the cut is performing a different comparison, and von Neumann showed that the predictions are the same wherever the cut is placed

along the chain of apparatus (1932, Chapter VI). What the standard formulation lacked was not a fixed location but the reason a cut must exist in every measurement; the structure of comparison supplies it.

Wigner's friend

Wigner (1961) asked what happens when a friend measures a system inside a sealed laboratory and Wigner, outside, treats the friend and the laboratory as one system. The pair of the subjective and the objective answers the question. The friend's comparison has the friend's apparatus as its subjective side and the system as its objective side; it has one definite outcome, in the friend's measurement time, and that outcome is laid out in the friend's record. Wigner's comparison has Wigner's apparatus as its subjective side and the whole laboratory, friend and record included, as its objective side. When Wigner measures the laboratory in a basis in which its records interfere, his comparison is a further comparison, with its own setting, and it too has one definite outcome. Each comparison has its own outcome, and no comparison has two.

Frauchiger and Renner (2018) sharpened the case. With two friends and two outside observers, three assumptions together lead to a contradiction: that quantum theory applies to agents who use it; that an agent may adopt as certain a conclusion that another agent has drawn with certainty; and that a measurement has a single outcome for the one who makes it. The structure of comparison keeps the first and the third, and it restricts the second. The laid-out record is shared, and the same for everyone who has the outcome (Section IV). An outside observer who measures a friend's laboratory in a basis in which its records interfere does not have the friend's outcome: his comparison takes the whole laboratory, the friend's record included, as its objective side, and does not find the friend's outcome. So a conclusion drawn within one comparison is a premise for every comparison that has its outcome, and for no comparison whose setting undoes that outcome. With the second assumption restricted in that way, the contradiction does not arise.

IV. Why Collapse Is Not Physical

In the standard formulation, the "collapse of the wavefunction" is treated as a physical process: an additional dynamical law that supplements unitary evolution. Treating collapse as a physical process is the source of the measurement problem: how can a physical process (collapse) violate another physical law (unitarity)?

In the IM framework, the state update upon measurement is not a physical process. It is not an additional dynamical law. It is an epistemic update: the comparison has occurred, and the result is now part of the omniscient content, the structural, knowable pattern that the system presents. Epistemic here means that what changes is the laid-out record, which is shared and the same for everyone who has the outcome; it does not mean a personal degree of belief. My account of the update parts from QBism (Section VIII) on that meaning of the word 'epistemic'.

The word has a second technical sense in the foundations of physics. In the classification of Harrigan and Spekkens (2010), a model is ψ-epistemic when the quantum state is knowledge of an underlying real state, and the theorem of Pusey, Barrett and Rudolph (2012) excludes such models when systems prepared independently have independent real states (preparation independence). The state of the derived formulation is not ψ-epistemic in that sense: there is no further state beneath it. No further state settles the outcome, since the residual belongs to comparison itself (Chapter 7, Section II); and no further state shifts the weights without settling the outcome, since a weight is the stable frequency of the same comparison, performed on the same state in the same context (Chapter 7, Section II), and the weight of a finding depends only on the subspace that represents it (Chapter 7, Section IV). In the terms of Chapter 3 the state exists: it is the laid-out record of the latest finding and of what a setting makes possible, the same for everyone who has the outcomes, and it is that record which a comparison updates.

The directed cycle of Axiom II provides the structural mechanism. Before the measurement, what has been found is laid out as structure (omniscient). The setting of the comparison, with its range of possible outcomes, is the stance from which the act proceeds (transcendent). The measurement is the act (immanent): an actualization, a transition from the possible to the actual. Its outcome becomes new laid-out structure (omniscient), which is the condition of the next setting. In a measurement, the directed cycle of Axiom II runs O → T → I → O, which is the cycle T → I → O → T of Chapter 4, Section IV, entered at the omniscient.

The directed cycle guarantees that this transition is structurally irreversible. The cycle O → T → I → O has a direction: it proceeds from omniscient to transcendent to immanent and back to omniscient. The immanent transition, the actualization of the measurement, cannot be reversed within the cycle. The irreversibility belongs to the passage between modalities, each step of which is many to one (Chapter 4, Section V), not to the operations within the immanent carrier. Once the comparison has occurred, the system has new omniscient content. The previous superposition is not "destroyed" by a physical process; it is superseded by the new content that the comparison has produced.

This account explains why the state update *appears* to be a physical process from within the standard formulation. The standard formulation does not have the content/context partition or the directed cycle as structural resources. It has only the Hilbert space and its operators. Within this impoverished framework, the transition from superposition to definite outcome can only be modeled as a dynamical change of state: a collapse. The collapse is an artifact of the framework's limitations, not a feature of the physics.

The outcome of a measurement is probabilistic for one reason, the reason Chapter 7, Section II gives: a description that fixed every finding would perfect symmetry and continuity together, which the Incommensuration Theorem excludes, so every lawful description leaves a residual that the law does not fix. The outcome itself is immanent and actual; that

is why Chapter 7 calls the probability of an outcome ontic in the structural sense. What is updated is the omniscient record of what has been found; that is why the update is epistemic. The two statements concern different modalities, and they do not conflict.

> **The Born rule at the seam between the two levels.**
>
> The Born rule stands at the seam between the two levels of the theory: the formalism is continuous, and the findings are discontinuously symmetric (Chapter 4, Section III). The Born rule carries a state of the formalism to the weights of findings. In the standard formulation the seam appears as the measurement problem; in the derived formulation it is the place where the Born rule acts. The rule cannot be derived from unitary dynamics alone, and it is the unique rule consistent with the algebra (Chapter 7).

V. The State After an Outcome

The standard formulation states, as Postulate 5, what the state becomes after an outcome: its projection onto the eigenspace of the outcome, renormalized. In the derived formulation this rule follows from two features of a comparison that has just been completed.

Let the outcome be k, with projection P_k, and let φ be the state before the comparison. The first feature is repeatability, which holds for a comparison of the first kind, the ideal measurement that Postulate 5 describes: such a comparison, repeated at once, returns the same outcome. Repeatability has its ground in what it is to be found again as the same thing: inertia is "the sameness of a pattern from one comparison to the next" (*Comparison; The Necessary Basis*, Chapter 8, Section I), so a thing once found, and changed by nothing since, is found again the same. The outcome has become new omniscient content, and the updated state φ_k

must give it weight one: $\varphi_k(P_k) = 1$. For a state, this means that φ_k sees only what lies within the outcome: $\varphi_k(a) = \varphi_k(P_k a P_k)$ for every element a.

The second feature is that the comparison changes nothing it did not compare. Stated as an identification: a comparison that leaves its context as it was leaves unchanged the relative weights of the state within the outcome, so that on the elements $P_k a P_k$, φ_k is proportional to φ. The two features, repeatability and the unchanged context, together fix the update:

$$\varphi_k(a) = \varphi(P_k a P_k) \,/\, \varphi(P_k).$$

The update that repeatability and the unchanged context fix is the projection rule of Lüders (1951). For a pure state $|\psi\rangle$ it gives the normalized projection $P_k|\psi\rangle \,/\, \|P_k|\psi\rangle\|$, which is Postulate 5 as stated in Chapter 1, degenerate case included. The rule is not an additional dynamical law. The Lüders rule states what the completion of a comparison means: the outcome is fixed (repeatability), and nothing outside the outcome is disturbed (the context is unchanged).

Most measurements are not of the first kind. A photon counter destroys the photon it finds, and an unsharp measurement does not separate its outcomes cleanly. Each is still an ideal comparison, made on a probe joined to the system. The measured system is joined to a probe, the joint system evolves (Postulate 3), and the probe alone is then compared by an ideal comparison (Ozawa, 1984). The Lüders rule applies to the probe. Tracing over the probe, which is the context that the comparison of the system does not register, leaves the system in the state

$$\varphi_k(a) = \varphi(K_k{}^* a K_k) \,/\, \varphi(K_k{}^* K_k),$$

where the operators K_k are fixed by the joining and by the probe's outcome k, the operators $K_k{}^*K_k$ are the effects of Busch's theorem (Chapter 7, Section III), and $\varphi(K_k{}^*K_k)$ are their weights. Tracing over the probe gives the general form of a measurement's effect on a state (Kraus, 1983). Postulate 5 is its ideal case, $K_k = P_k$.

The Running Example: Three Magnets in a Row

Send the silver atom through a magnet along z, and keep only the atoms that land up. By the Lüders rule their state is now up along z. Send these through a magnet along x: half land up and half land down. Keep those that land up along x, and send them through a magnet along z again: once more half land up and half land down. The record of the first finding is still on the first screen, but the state, updated by the second comparison, is up along x, and a state up along x gives one half to each outcome along z. Each magnet is a comparison with its own setting, and its outcome replaces the state the next comparison meets. The cut lies between each magnet with its screen and the atom; a camera reading the first screen would be a further comparison, with the screen as its objective side.

VI. Decoherence from Containment

Decoherence (the process by which a quantum system loses its coherent superposition through interaction with its environment) is widely regarded (Zurek, 2003) as the mechanism that produces the appearance of classical behavior from quantum dynamics.

In the IM framework, decoherence is a structural consequence of the content/context partition.

A system that is contained within a larger environment cannot maintain coherence independently. The system's content is entangled with the environment's content. The off-diagonal elements of the system's density matrix, the coherence terms, are distributed across the system-environment entanglement. They are not destroyed; they become inaccessible to any measurement that probes the system alone.

In IM terms: the content/context boundary between the system and its environment is a comparison boundary. As the environment interacts with the system, the content of the comparison spreads across the joint system. The coherence terms become part of the joint content and are no longer attributable to the subsystem alone.

The spreading of the content across the joint system is decoherence, and not collapse. Decoherence does not produce a definite outcome. It produces an *effective* mixture: a density matrix that is diagonal in the preferred basis. The preferred basis is the setting of the environment's own comparison of the system: the environment compares the system in the respects its interaction distinguishes, and those respects are the pointer basis (Zurek, 2003). The definite outcome requires the additional step of the immanent transition: the actualization described in the preceding section.

The structural distinction between decoherence and collapse is important. Decoherence is the outward movement of the content/context boundary: as the environment interacts with the system, the joint system becomes the content, and the coherence terms belong to it and not to the subsystem. Collapse, identified in the Immanent Metaphysics with the immanent actualization, is the establishment of new omniscient content through a specific comparison. Decoherence prepares the conditions for the measurement (by selecting a preferred basis); the immanent transition completes the measurement (by actualizing one outcome).

VII. The Dissolution

The measurement problem dissolves when quantum mechanics is derived from the structure of comparison rather than assembled from postulates.

In the derived formulation, the pair of the subjective and the objective is present from the start. Every comparison has a built-in distinction between the side from which it is made (the subjective, the apparatus)

and the side about which it is made (the objective, the system). Every comparison has a Heisenberg cut, and the comparison performed sets where it lies; no cut is an arbitrary convention.

The state update upon measurement is not a physical process that must be reconciled with unitary evolution. It is an epistemic transition within the directed cycle O → T → I → O. The two kinds of time, evolution time (transcendent, continuous) and measurement time (immanent, discrete), are structurally distinct (Chapter 7, Section II). They are not two descriptions of the same process; they are two different modalities. The demand that one be derived from the other is the category error that produces the measurement problem.

The derived formulation does not begin with classical concepts and then accommodate quantum ones with difficulty. It begins with the structure of comparison, from which both classical and quantum behavior emerge. The classical limit has two parts, and neither needs an interpretive commitment. Decoherence (Section VI) moves the content/context boundary outward, so that coherence belongs to the joint system and not to the part. The immanent act then gives one outcome (Section IV). What remains of the part can then be described classically, and a classical description of matter is content described in the mode pair of context, smooth findings with a symmetric formalism, valid within the limits that physics has long known (*Measurement*, Chapter 9, Section VIII). The quantum description is the foundational description of the same content, with discrete, exact findings and a continuous formalism, and the classical description is its limit: "The quantum theory is the foundational form of the field, and the classical theory is its limit" (*Measurement*, Chapter 9, Section VII). The classical limit is a passage between the two mode pairs of one content, not a reduction of one theory to another.

That the classical description is the limit of the quantum one shows in the evolution of Postulate 3 itself. A path is one way a body can go from one place and time to another, and its action S, divided by ħ, is the phase the body gathers along that path. Divided into short steps of time, the evolution $U(t) = \exp(-iHt/\hbar)$ gives the amplitude to go from the one

place and time to the other as the sum, over every path between them, of the phase factor $\exp(iS/\hbar)$ (Dirac, 1933; Feynman, 1948). Reached in this way, the sum over paths follows from the derived evolution; it is not taken as a translation of quantum mechanics (Chapter 2, Section I). For a body whose action is many times $\hbar$, a small change of path changes the phase by many whole cycles, so the phases of neighboring paths differ widely and cancel, except near the path whose action does not change to first order when the path is changed a little, the path of stationary action, where neighboring paths share nearly one phase and add. So such a body is found along the path of stationary action. The equations of that path, the equations of Euler and Lagrange, are Newton's laws of motion, and the mean values of the body's position and momentum follow the classical equations (Ehrenfest, 1927). Where the action is only a few times $\hbar$, as for small things over short times, neighboring paths do not cancel, and interference shows. Where a body is kept from its surroundings, as an isolated body kept cold is, decoherence (Section VI) does not set in, and its alternatives keep their interference however large the body is. Where nearby paths of stationary action diverge from one another quickly, as in a chaotic system, the spread of the body grows from the scale set by $\hbar$ to the scale of its motion after a time that grows only as the logarithm of $S/\hbar$, divided by the rate at which the paths diverge (Berman and Zaslavsky, 1978), and after that time the single path fails.

The measurement problem is a symptom of starting from the wrong place. The measurement problem is an artifact of assembling the postulates from multiple, structurally incompatible sources and then demanding that they form a consistent whole. They do not. The IM derivation replaces them with a single consistent source, and the symptom disappears.

What the dissolution leaves is the residual, and the residual is no contradiction. *Measurement* states it: "For quantum mechanics the residual is the measurement problem: the discontinuity of a definite finding, which a continuous formalism does not contain." The continuous formalism approaches every finding as closely as any measurement can resolve, and it never contains the finding itself: the selection of one outcome from among

its possibilities is given in the act of comparison and not in the formalism (*Measurement*, Chapter 8, Section VII). The theorem guarantees that gap, and no future theory removes it, since a formalism that contained its findings exactly would perfect both characters in one mode. So the contradiction among the postulates dissolves, and the residual remains as what it is: the place where the formalism, which exists, meets the finding, which is real, and where the Born rule stands.

VIII. Comparison with Interpretations

The major interpretations of quantum mechanics can be understood, within the IM framework, as strategies for coping with the measurement problem that differ in which modality they privilege and which they suppress. *Two Theories, One World* and *The Unquantized Arena* treat the interpretations in the same way, as modal strategies.

Copenhagen. The Copenhagen interpretation preserves the immanent modality: measurement is primitive; the measurement event is real and not further analyzable. This respects the IM structure (Axiom I: the immanent is more fundamental). But Copenhagen leaves the three-time structure (evolution, measurement and geometric time, Chapter 7, Section II) unacknowledged. It does not explain why the classical/quantum boundary exists or where the Heisenberg cut should be placed. Copenhagen takes the preparation and the choice of setting as classical givens and gives no account of the range of possibilities they open; it is silent on the transcendent, and that silence is why it cannot say where the cut lies. The IM framework agrees with Copenhagen's core commitment (measurement is primitive) but provides the structural reason for that commitment: the immanent modality is the modality of actualization, and every comparison has a cut, which the comparison performed sets.

Many-worlds. The many-worlds interpretation (Everett, 1957) eliminates measurement time entirely. There is no collapse; only unitary evolution and geometric structure. In IM terms, this abolishes the immanent

modality, violating Axiom III (three modalities: distinct, inseparable, non-interchangeable). Its characteristic problem (the inability to derive the Born rule from within the formalism) is the structural consequence of dropping the immanent modality. The Born rule is the bridge from the formalism to the findings, and the findings are given in the immanent act; remove the modality, lose the bridge.

Bohmian mechanics. Bohmian mechanics (Bohm, 1952) adds a hidden trajectory to every particle, embedding measurement time within geometric time. In IM terms, this reduces the immanent to the omniscient, violating non-interchangeability. Its characteristic problem, a preferred foliation of spacetime that its guidance equation requires and no experiment detects, is the structural consequence of forcing two distinct time concepts into one.

QBism. QBism (Fuchs and Schack, 2013) treats evolution time as the agent's expectation: the quantum state is an agent's personal belief, not a feature of the world. In IM terms, this eliminates the transcendent modality as an ontological category. Its characteristic problem (the inability to account for the objective regularities that quantum mechanics predicts) is the structural consequence of removing the transcendent modality from the domain's structure.

Decoherence and consistent histories. The decoherence program provides the mechanism by which classicality emerges from quantum dynamics. The IM framework agrees with the mechanism but provides the structural reason for it: decoherence is the outward movement of the content/context boundary through interaction with a larger context, so that the coherence terms belong to the joint system. The consistent (or decoherent) histories approach (Griffiths, 1984) rejects a single preferred history and works with families of consistent decoherent histories. In IM terms, the consistency condition corresponds to the inseparability clause of Axiom III: a history joins settings, acts and findings, and a family that separates them assigns no probabilities.

Objective collapse. Objective-collapse theories (Ghirardi, Rimini and Weber, 1986; Penrose, 1996) modify the evolution, by a stochastic term (Ghirardi, Rimini and Weber) or by a reduction that gravity induces (Penrose), so that a superposition of macroscopically different states collapses of itself. In IM terms, this folds the immanent into the transcendent: the act becomes a term of the evolution, and measurement time is made part of evolution time, violating non-interchangeability. Its characteristic problem, new physics that has not been observed, is the structural consequence of making the act a law of evolution. Objective collapse is the mirror image of Bohmian mechanics, which folds the immanent into the omniscient.

Relational quantum mechanics. Relational quantum mechanics (Rovelli, 1996) holds that the outcome of an interaction is a fact relative to the system that interacts, and in its original form it holds no record of facts shared by all. In IM terms it keeps the immanent, the act and its perspective, much as the pair of the subjective and the objective does, and the transcendent, the setting of each interaction; it suppresses the omniscient as shared, the laid-out record that is the same for everyone who has the outcome. Its characteristic problem, saying how the facts relative to different systems agree, is the structural consequence of that suppression. Its later statements add links between the perspectives of different systems, which restore a record shared across them (Adlam and Rovelli, 2023).

The modal map of the interpretations.

The interpretations can be arranged by which of the three modalities each keeps and which it suppresses. The decoherence program and consistent histories have no row of their own: the paragraph above places them as a mechanism and a consistency condition that the IM framework keeps, not as a choice of a modality to suppress.

Interpretation	Keeps	Suppresses
Copenhagen	The immanent (measurement is primitive) and the omniscient	Silent on the transcendent
Many-worlds	The transcendent (evolution) and the omniscient (geometry)	Removes the immanent
Bohmian mechanics	The omniscient (trajectories in geometric time)	Folds the immanent into the omniscient
QBism	The immanent (the agent's act) and the omniscient (the formalism as coherence)	Denies ontological status to the transcendent
Objective collapse	The transcendent (evolution, with a collapse term) and the omniscient	Folds the immanent into the transcendent
Relational quantum mechanics	The immanent (the act and its perspective) and the transcendent (the setting of each interaction)	Suppresses the omniscient as shared

Each of the three modalities is suppressed by some interpretation: the immanent by many-worlds, and folded into another modality by Bohmian mechanics and by objective collapse; the transcendent by QBism, and passed over in silence by Copenhagen; the omniscient, as shared, by relational quantum mechanics. The IM framework is distinguished by acknowledging all three and by deriving their relationships from the axioms. The characteristic failure mode of each interpretation corresponds to the modality it suppresses.

IX. Summary

The measurement problem has generated a century of debate because it is a genuine inconsistency in the postulational framework of quantum mechanics. The postulates were assembled from different sources and encode incompatible assumptions about the nature of measurement.

The Immanent Metaphysics does not resolve the inconsistency. It dissolves it. The derivation begins with the structure of comparison, which has the pair of the subjective and the objective and the directed cycle as built-in structural features. These features provide the system/apparatus distinction, the measurement irreversibility, and the state that follows an outcome, which the standard formulation lacks.

The measurement problem was never a problem of physics. It was a problem of starting point. Start from comparison, and the contradiction does not arise; what remains is the residual, the place where the Born rule stands, which the theorem guarantees and no future theory removes.

KEY CONCEPTS IN THIS CHAPTER

- **Measurement problem**: The inconsistency between unitary evolution (Postulate 3) and collapse (Postulates 5 and 6) in the standard formulation of quantum mechanics.
- **Subjective and objective**: The structural division of every comparison into the side from which it is made (the apparatus) and the side about which it is made (the system). Present in every comparison and found by perspective, it provides the system/apparatus distinction without an arbitrary Heisenberg cut.
- **Residual**: What every lawful description leaves unfixed; for quantum mechanics, the definite finding, which the continuous formalism does not contain (*Measurement*, Proposition 8.2).
- **Directed cycle**: The cycle T → I → O → T of the three modalities (Axiom II), entered at the omniscient as O → T → I → O. Provides structural irreversibility for the measurement transition without an additional dynamical law.
- **Immanent transition**: The passage from possible to actual in the directed cycle. The structural analogue of the "collapse" in the standard formulation, but not a physical process.
- **State update**: The projection rule of Lüders, $\varphi_{\mathrm{k}}(a) = \varphi(P_{\mathrm{k}} a P_{\mathrm{k}}) \,/\, \varphi(P_{\mathrm{k}})$, derived from the repeatability of a comparison of the first kind and from the identification that a comparison which leaves its context unchanged leaves the relative weights of the state unchanged.
- **Born rule at the seam**: The Born rule carries a state of the continuous formalism to the weights of discontinuously symmetric findings; it stands at the seam between the two levels of the theory (Chapter 4, Section III).
- **Heisenberg cut**: The boundary between system and apparatus in the standard formulation. Arbitrary in the standard framework; in the IM framework, present in every comparison and set by the comparison performed.

- **Decoherence**: The loss of quantum coherence through interaction with the environment. The outward movement of the content/context boundary through entanglement with a larger context, so that the coherence terms belong to the joint system. Not the same as collapse.
- **Modal map of the interpretations**: The arrangement of interpretations by which modality they suppress: Copenhagen (silent on the transcendent, and so unable to place the cut), many-worlds (suppresses immanent), Bohmian (reduces immanent to omniscient), QBism (denies ontological transcendent), objective collapse (folds immanent into transcendent), relational quantum mechanics (suppresses the omniscient as shared). Each interpretation's failure mode corresponds to its suppressed modality.
- **Wigner's friend**: The friend's comparison and Wigner's are different comparisons, with different subjective sides, and each has one outcome; a conclusion drawn within one comparison is a premise for every comparison that has its outcome (Section III).
- **General measurement**: An ideal comparison of a probe joined to the system; tracing over the probe gives $\varphi_{K}(a) = \varphi(K^{*}aK) \,/\, \varphi(K^{*}K)$, with the Lüders rule as its ideal case (Section V).
- **Two senses of the measurement problem**: The contradiction among the postulates, which the derivation dissolves, and the residual between a continuous formalism and a definite finding, which remains as no contradiction (Section I).

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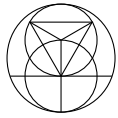
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PART IV

The Wider Landscape



Chapter 10

Quantum Information from the Comparison Algebra

Qubits, Gates, and the No-Cloning Theorem

ABSTRACT

The comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ encodes within its structure the fundamental objects of quantum information theory. The qubit, the elementary unit of quantum information, is the module $\mathbb{C}^2$ of the quaternionic carrier $\mathbb{H}$, its smallest faithful complex module, whose ray space is the Bloch sphere S^2. Single-qubit gates are the unitaries of that module: up to a phase that no comparison detects, they are the unit quaternions $SU(2)$, which act on the Bloch sphere as the rotations $SO(3)$. The composite algebra $\mathbb{H} \otimes \mathbb{H}$ is isomorphic to $M_4(\mathbb{R})$, and its unitaries create no entanglement. The entangling gates, the controlled-NOT among them, are unitaries of the complex space $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose operators form $M_4(\mathbb{C})$; on that space the gate group is $U(4)$, $SU(4)$ up to a phase. The complex structure of that space is the faithful complex context of evaluation that Chapter 5, Section III identifies. The no-cloning theorem follows from the linearity of the derived evolution on the Hilbert space, and the structure of comparison marks which states can be copied: the outcomes of one setting can be copied, and no state that lies between them can be.

I. The Qubit from the Immanent Modality

The qubit, the fundamental unit of quantum information, is conventionally defined as a two-level quantum system whose state space is a two-dimensional complex Hilbert space $\mathbb{C}^2$. The definition is operational: a qubit is whatever has two perfectly distinguishable states and a continuum of superpositions between them. The question that the existing axiomatic programs leave unanswered is why such a system should exist at all. The operational definition presupposes, without explaining why, that nature furnishes two-level systems.

The Immanent Metaphysics provides an answer. The comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is derived in Chapter 5. In the focus of the two laws (Chapter 5, Section VI), each of its summands carries one of the three modalities: $\mathbb{C}$ the omniscient, an invariant label laid out as structure; $\mathbb{H}$ the immanent, the carrier of process, in which every non-zero element is invertible; and $M_3(\mathbb{C})$ the transcendent, with its range $\mathbb{C}P^2$ of possible states.

The qubit lives in the quaternionic sector. The algebra $\mathbb{H}$ acts faithfully on the two-dimensional complex vector space $\mathbb{C}^2$, through the representation of Chapter 8, Section IV, which identifies the quaternionic units i, j, k with $i\sigma_z$, $i\sigma_y$, $i\sigma_x$, where σ_x, σ_y, σ_z are the Pauli matrices. As a real algebra, $\mathbb{H}$ has only real numbers as observables; its non-commutativity shows in its unitaries, the unit quaternions, which form $SU(2)$. Over $\mathbb{C}$ it generates all operators on its module, $M_2(\mathbb{C})$, and these are the observables of the qubit. The complex lines in $\mathbb{C}^2$ form the ray space of $\mathbb{H}$, the sphere $\mathbb{C}P^1 = S^2$: the Bloch sphere of quantum information theory.

Chapter 8, Section IV states the Bloch sphere in full, as the ray space of the immanent carrier: each of its points is a ray of the module of $\mathbb{H}$, the two poles of any axis are two perfectly distinguishable states, and the unit quaternions rotate among the points.

The two-dimensionality of the qubit

In the conventional framework, the qubit's two-dimensionality is an empirical fact elevated to a definition. In the IM framework, it is a derived consequence: the Three-Carrier Theorem gives the immanent modality the one non-commutative division algebra, which the Frobenius theorem shows to be $\mathbb{H}$, and $\mathbb{H}$ acts faithfully on $\mathbb{C}^2$ and on no lower-dimensional complex space. The qubit is the smallest system that can carry the full content of the immanent modality.

In physics the same module $\mathbb{C}^2$ appears twice, once in each mode of the ICT (Chapter 8, Section IV): as weak isospin in the internal space, and as spin in spacetime. The spin of an electron is the direct case of the second. Other experimental qubits (photon polarization, two levels of an atom or circuit) carry the same module $\mathbb{C}^2$ as their state space without being spins; what the derivation fixes is the module they share.

The Bloch ball, whose surface is the sphere S^2 of rays, has three real dimensions, matching the three quaternionic units i, j, k. Masanes, Müller, Augusiak and Pérez-García (2014) showed that the three-dimensionality of the Bloch ball follows from entanglement combined with continuous reversibility. The IM derivation arrives at the same result from a different direction, not from information-processing requirements but from the algebraic structure of comparison itself.

II. Quantum Gates from the Comparison Algebra

A quantum gate is a unitary transformation acting on a Hilbert space. In the standard formalism, the set of available gates is determined by engineering capability: any unitary is in principle realizable, and the question is which finite set is sufficient to approximate all others. The IM framework provides a structural account of why the gate groups take the specific forms they do.

The unitary groups of the sectors of A_F are the groups of transformations that act on each sector's Hilbert space while preserving its structure. For the $\mathbb{C}$ sector this is $U(1)$, a phase. For the $\mathbb{H}$ sector it is the group of unit quaternions, $SU(2)$, acting on $\mathbb{C}^2$; acting by conjugation on $\mathbb{H}$ itself, the same group gives the rotations $SO(3)$ of the Bloch sphere. For $M_3(\mathbb{C})$ it is $U(3)$.

Single-qubit gates are the unitaries of $\mathbb{C}^2$; up to a phase that no comparison detects, they are the unit quaternions $SU(2)$, which act on the Bloch sphere as the rotations $SO(3)$.

Two-qubit gates require the tensor product. The composite algebra for two quaternionic sectors is $\mathbb{H} \otimes \mathbb{H}$, which is isomorphic to $M_4(\mathbb{R})$ as a real algebra. Its unitaries are the local gates $u \otimes v$ and their products with the exchange of the two qubits, and none of them creates entanglement. The entangling gates, CNOT among them, are unitaries of the complex space $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose operators form $M_4(\mathbb{C})$, the complexification of $M_4(\mathbb{R})$. The complex structure of that space is the faithful complex context of evaluation that Chapter 5, Section III identifies. The gate group of the composite system is $U(4)$, $SU(4)$ up to a phase, acting on the four-dimensional Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$. The controlled-NOT gate and the controlled-phase gate are elements of $U(4)$ (of $SU(4)$ up to a phase) that generate the full group when combined with single-qubit gates (Barenco and colleagues, 1995). The Solovay-Kitaev theorem guarantees that any element of $SU(4)$ can be approximated to arbitrary accuracy by a sequence of gates drawn from a finite set, closed under inverses, that generates a dense subgroup; for an error ε, the length of the sequence grows only as a power of $\log(1/\varepsilon)$ (Dawson and Nielsen, 2006).

The CNOT gate and entanglement creation.

The controlled-NOT gate, acting on the computational basis as $|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, $|11\rangle \to |10\rangle$, is the simplest two-qubit gate that creates entanglement from separable states. Entanglement creation is the composition of two comparison relations (Chapter 8, Section II): the joint system enters a state that is not decomposable into independent comparisons of its subsystems. The CNOT is a specific unitary of the complexified composite algebra $M_4(\mathbb{C})$ that establishes this irreducible joint comparison.

The universality of the gate set {H, T, CNOT} (the Hadamard, T-gate, and controlled-NOT) is a standard result in quantum information theory (Nielsen and Chuang, 2000, Chapter 4). The IM contribution is not a new derivation of this universality but a structural account of why the gates are drawn from $U(2)$ and $U(4)$ rather than from some other group. The single-qubit group is, up to a phase, the unit group of the quaternionic carrier, which the three axioms and the Frobenius theorem fix. The two-qubit group is the unitary group of the complex composite space $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose complex structure is the identification of Chapter 5, Section III.

III. The No-Cloning Theorem

The no-cloning theorem (the impossibility of creating an identical copy of an arbitrary quantum state) is one of the foundational results of quantum information theory. In the standard formalism, it follows from the linearity of quantum mechanics: a unitary transformation that maps $|\psi\rangle|0\rangle \to |\psi\rangle|\psi\rangle$ for all $|\psi\rangle$ would have to be both linear and state-duplicating, and these requirements are incompatible.

In the IM framework, the no-cloning theorem requires no additional input beyond the structures already derived. The GNS construction on A_F produces a Hilbert space H_F with a linear representation π of the algebra.

Every operator $\pi(a)$ is a linear map of H_F, and so is every unitary evolution on H_F (Chapter 6, Section VI). The no-cloning result then follows by the standard argument (Wootters and Zurek, 1982).

The structure of comparison also says which states cannot be copied. A comparison is a relation between domains: the six intrinsics (sameness, difference, content, context, subjective, objective) constitute the relational structure. What cannot be copied is a state that is not one of a set of perfectly distinguishable states. Copying requires a comparison that finds the state, and a comparison finds only the outcomes of its own setting; a state that lies between those outcomes is changed into one of them by the finding (Chapter 9, Section V). States that are outcomes of one setting can be copied, as the CNOT copies $|0\rangle$ and $|1\rangle$, which is why classical records can be copied. No-cloning is the information-theoretic expression of the fact that comparison is constitutive, and not only descriptive.

The Running Example: What Can Be Copied

After a magnet along z, the silver atom of Chapter 3 is up or down along z, and the spot on the screen records which. That outcome can be copied as often as one likes: a second qubit, prepared "up" and joined to the atom by a controlled-NOT gate that flips it when the atom is down, is left in the same state along z as the atom, and so do cameras photographing the screen. An atom prepared up along x cannot be copied. A magnet along z, asked to find it, would find up or down along z, each with weight one half, and the state up along x would be replaced by the outcome found. The atom's state between the outcomes of one setting is what no comparison of that setting can carry away.

Cloning and the six intrinsics.

The impossibility of cloning can be connected directly to the content/context distinction among the six intrinsics. A cloning operation for an arbitrary state would need to treat the quantum state purely as content: something to be read and copied. But the state is also context: it defines the framework within which measurement outcomes are possible. Content and context cannot be separated (the inseparability of the pair, Chapter 3, Section III), and the attempt to extract pure content for duplication necessarily disturbs the contextual aspect. The inseparability of content and context is the same feature that gives the form to the state after an outcome (Chapter 9, Section V).

IV. Summary

The comparison algebra A_F encodes the fundamental objects of quantum information theory. The qubit is the module $\mathbb{C}^2$ of the quaternionic carrier, the smallest faithful complex module of $\mathbb{H}$, with the Bloch sphere as its ray space; the spin of an electron carries it directly, and other experimental qubits carry it as their state space. Single-qubit gates are the unit quaternions, up to a phase. The composite algebra $\mathbb{H} \otimes \mathbb{H}$ alone gives only local gates and the exchange of the two qubits; the entangling gates act on the complex composite space $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose complex structure is the context of evaluation that Chapter 5, Section III identifies.

The no-cloning theorem is a consequence of the linearity of the derived evolution on the Hilbert space. The structure of comparison marks which states can be copied: the outcomes of one setting can be copied, and a state that lies between them cannot be, because a comparison finds only the outcomes of its own setting.

The objects of quantum information are therefore not conventions of engineering. They are the module, the unit group and the composite space of the comparison algebra, stated in the language of computation.

KEY CONCEPTS IN THIS CHAPTER

- **Qubit from $\mathbb{H}$**: The qubit is the module $\mathbb{C}^2$ of the quaternionic carrier of A_F, its smallest faithful complex module, with the Bloch sphere S^2 as its ray space. The spin of an electron carries this module directly; other experimental qubits carry it as their state space.
- **Gate groups from the algebra**: Single-qubit gates are the unitaries of $\mathbb{C}^2$, the unit quaternions $SU(2)$ up to a phase. The composite algebra $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$ gives only local gates and the exchange; the entangling gates are unitaries of the complex space $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose operators form $M_4(\mathbb{C})$, with gate group $U(4)$, $SU(4)$ up to a phase. These gate groups are derived from the algebra, not assumed from engineering convention.
- **No-cloning**: The impossibility of copying a state that is not one of a set of perfectly distinguishable states. It follows from the linearity of the derived evolution on H_F; the outcomes of one setting can be copied.

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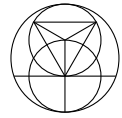
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Chapter 11

What Other Programs Assume and the Immanent Metaphysics Derives

A Comparative Inventory

ABSTRACT

The preceding chapters have presented the Immanent Metaphysics derivation of quantum mechanics from the structure of comparison. This chapter places that derivation among the existing programs: the reconstructions of Hardy (2001), Chiribella, D'Ariano, and Perinotti (2011), and Masanes and Müller (2011); QBism (Fuchs and Schack); constructor theory (Deutsch and Marletto); and the route of non-commutative geometry by which Chamseddine and Connes (2007) select the algebra of the Standard Model. For each of six features of quantum mechanics (complex Hilbert space structure, the Born rule, superposition, entanglement, the gauge group of the Standard Model, and the particle content) it asks whether the program assumes the feature or derives it. The reconstruction programs derive the general framework from operational premises they take as given; none reaches the specific algebra. Chamseddine and Connes reach the specific algebra from conditions on a spectral triple that they take as given. The IM derives the specific algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from the structure of comparison, and with it reasons for the conditions it shares with the route of non-commutative geometry (KO-dimension 6, the three carriers and their types, the rank three, and a meaning in comparison for the order-one condition); and it does not use the one condition it does not share, the Majorana coupling.

I. The Comparison Table

Two tables compress the full analysis of this chapter. The features compared are features of quantum mechanics: structural elements that appear in the standard Hilbert space formalism. In the first table they head the columns, and in the second they head the rows. The first table takes the five reconstruction and interpretive programs: Hardy; CDP, the program of Chiribella, D'Ariano and Perinotti (Section III); Masanes and Müller; QBism; and constructor theory. Müller (2021) reviews the reconstruction

programs among them. The entries are: *D* for "derived from the program's axioms," *A* for "assumed as an axiom or presupposition," and a centered dot (·) for "not addressed." Where a derivation is partial or carries a caveat, the entry is annotated. None of the five addresses the gauge group or the particle content, so those two features are left out of the first table.

Program	Hilbert space	Born rule	Superposition	Entanglement
Hardy	D (finite dim.)	D	D	D
CDP	D (finite dim.)	D	D	D
Masanes and Müller	D (finite dim.)	D	D	D
QBism	A	A (reinterpreted)	A	A
Constructor theory	·	D (decision-theoretic)	·	D (partial)

The second table sets the two routes that reach the specific algebra side by side: NCG, the route of non-commutative geometry taken by Chamseddine and Connes (Section VII), and the Immanent Metaphysics. Here "Wider program" means "derived beyond this book."

Feature	Non-commutative geometry	Immanent Metaphysics
Hilbert space	A	D
Born rule	A	D
Superposition	A	D
Entanglement	A	D

Feature	Non-commutative geometry	Immanent Metaphysics
Gauge group	D (from its conditions)	D (with unimodularity, Chapter 5)
Particle content	D (one generation)	Wider program

The pattern is stark. All three probabilistic programs (GPT and OPT: Hardy, CDP, Masanes and Müller) derive the same cluster of results: complex Hilbert spaces, the Born rule, superposition, and entanglement, all in finite dimensions. QBism is an interpretation; it takes the full formalism as given and reinterprets probability. Constructor theory operates at a meta-level, characterizing which tasks are possible rather than deriving the specific formalism. IM derives the first five in this book: the four that the probabilistic programs also derive, plus the gauge group; the particle content belongs to the wider program. The route of non-commutative geometry taken by Chamseddine and Connes reaches the gauge group and the particle content of one generation from conditions of its own (Section VII).

The remainder of this chapter examines each program in turn, identifying what it assumes, what it derives, and what IM can supply that the program cannot.

II. Hardy's Five Reasonable Axioms

Hardy's 2001 paper is the founding document of the operational reconstruction program. Working within the framework of generalized probabilistic theories (GPTs), Hardy shows that complex quantum theory in finite dimensions is the unique theory satisfying five axioms: probabilities, simplicity, subspaces, composite systems (tomographic locality), and continuity (continuous reversible transformations).

The derivation is elegant and influential. From five physical requirements Hardy obtains the density-matrix formalism, the trace rule for probabilities, unitary evolution, and the tensor product rule for composites. Dropping the continuity axiom (or just the word "continuous") gives classical probability theory instead: a clean structural bifurcation between classical and quantum.

What Hardy assumes is the entire probabilistic-operational framework. Systems, states, preparations, measurements, and probabilities are all primitive notions. The existence of distinguishable states (the assumption that there exist measurements perfectly discriminating certain states) is taken as given. Tomographic locality (the assumption that local measurements suffice to determine the state of a composite system) is an axiom, not derived. The simplicity axiom (that the number of degrees of freedom K takes the smallest value consistent with the other axioms) is the most *ad hoc* of the five, a selection principle rather than a physical constraint.

The IM framework derives four of Hardy's axioms from the three axioms and the ICT, and makes the fifth, simplicity, unnecessary.

Hardy's Axiom 1 (probabilities) is derived as Chapter 7, Section II derives it. The Incommensuration Theorem leaves a residual in every lawful description: the state does not fix the finding. Sameness applied to repetition gives each finding one stable relative frequency, its probability. The probability is structural; it does not measure anything an observer lacks.

Hardy's Axiom 2 (simplicity) is made unnecessary by the comparison algebra. Hardy's axioms give $K = N^r$, where K is the number of real parameters of a state and N the number of its perfectly distinguishable states; simplicity takes the smallest r the other axioms allow, and continuity excludes $r = 1$, so $K = N^2$ ($N^2 - 1$ for normalized states). In IM the state spaces are those of the complex matrix algebras of Chapter 6, on which an N-level system has N^2 real parameters, so there is no family to select from. The simplicity axiom is not needed: there is nothing from which to select the simplest.

Hardy's Axiom 3 (subspaces) follows from the corner algebras of the comparison algebra. A system whose state is confined to the range of a projection p behaves as the corner algebra pAp prescribes. For a carrier $M_n(\mathbb{C})$ and a projection of rank m, pAp is $M_m(\mathbb{C})$: an m-dimensional subspace of an n-level system behaves as an m-level system.

Hardy's Axiom 4 (tomographic locality) is derived from the complex tensor product. The observables of a composite are the self-adjoint elements of the complex algebra $A_1 \otimes A_2$ (Chapter 6), and a complex tensor product is spanned by the products $a_1 \otimes a_2$. A state of the composite is therefore fixed by its values on products, which are the joint statistics of local measurements: this is tomographic locality. It holds because the observables form complex algebras, which the faithful complex context of evaluation secures (Chapter 5, Section III); for real matrix algebras the symmetric elements of a product are not spanned by products of symmetric elements, and real quantum theory fails tomographic locality (Hardy and Wootters, 2012).

Hardy's Axiom 5 (continuity) is derived from the unit groups of the carriers. The unit group of each non-commutative carrier acts continuously and transitively on its rays: SU(2) on the sphere S^2 of the module of $\mathbb{H}$, and U(3) on $\mathbb{C}P^2$ (Chapter 5, Section VIII). Any pure state of such a module is joined to any other by a continuous path of reversible transformations. This continuum belongs to the formalism, which rests on continuous structures, while what is found is discrete (Chapter 4, Section III).

The scope difference.

Hardy derives finite-dimensional QM; IM derives the finite algebra A_F and, with it, the product triple of Chapter 6, whose infinite-dimensional factor $L^2(M, S)$ comes from the spacetime mode; and this book stops at the masses and mixings. Hardy gives no account of which specific quantum theory obtains in the actual world; IM derives the specific algebra A_F; the wider program carries the particle spectrum. Hardy's axioms select quantum theory from among the generalized probabilistic theories; IM's axioms select the specific quantum theory, the one with the Standard Model gauge group $U(1) \times SU(2) \times SU(3)$ (given the unimodularity condition of Chapter 5), from among all possible quantum theories.

III. Chiribella, D'Ariano, and Perinotti

The CDP program (2011) works in the framework of operational-probabilistic theories (OPTs), which is more general than Hardy's GPT framework. Their six axioms (five axioms plus the purification postulate) derive the same core results as Hardy (complex Hilbert spaces, density matrices, unitary channels, the tensor product rule) plus additional structural features: no-cloning and teleportation.

The five axioms are causality (no future-to-past signaling), perfect distinguishability (non-completely-mixed states admit at least one excluding measurement), ideal compression (every state admits an ideal encoding into a minimal-size system), local distinguishability (bipartite states differing globally can be distinguished locally), and pure conditioning (a bipartite pure state measured on one side yields pure conditional states on the other; their later textbook, D'Ariano, Chiribella and Perinotti, 2017, uses in its place atomicity of composition: the sequential composition of two atomic transformations is atomic).

The purification postulate does the heaviest lifting. It states that every mixed state of a system can be obtained as the marginal of a pure state of a larger system, and that this purification is unique up to reversible transformations on the purifying system. CDP characterize quantum theory as "the only pure and reversible theory of information": a slogan whose content is that purification, with the other five principles, selects QM uniquely from among all operational-probabilistic theories.

What CDP assume includes: the operational-probabilistic framework itself, systems, tests, events, composition rules, and probabilities as primitive notions; causality (no retrocausal signaling) as an axiom; local distinguishability (tomographic locality) as an axiom; and the purification postulate as a fundamental principle not derived from anything deeper.

The IM framework derives the purification postulate as a theorem. Once Chapter 6 has given the Hilbert space and the tensor product, every density matrix ρ on $\mathbb{C}^n$ is the marginal of a pure state on $\mathbb{C}^n \otimes \mathbb{C}^n$, and any two purifications on one purifying system differ by a unitary on that system. The GNS construction of Chapter 6 gives this purification directly. For a faithful state $a \mapsto \mathrm{tr}(\rho a)$ on $M_n(\mathbb{C})$, the GNS space is $M_n(\mathbb{C})$ itself with the inner product $\langle a, b\rangle = \mathrm{tr}(\rho a^* b)$; the map $a \mapsto a\rho^{1/2}$ carries it onto the $n \times n$ matrices with the trace inner product, which is $\mathbb{C}^n \otimes \mathbb{C}^n$. The cyclic vector, the identity of the algebra, goes to $\rho^{1/2}$, and its marginal on the first factor is ρ. In finite dimension the GNS construction and purification are one construction. Identified in the Immanent Metaphysics, the purifying system is the context that an incomplete comparison traced over: content and context are inseparable (Chapter 3, Section III).

CDP's causality axiom (the requirement that the probability of a preparation is independent of the choice of subsequent measurement) is derived from Axiom II (the cycle of precedence), once preparation, test and outcome are bound to the modalities. A preparation is the stance from which a test proceeds (transcendent), the test is the act (immanent), and its outcome is the finding laid out (omniscient). Axiom II puts a class of the transcendent before an instance of the immanent: the preparation,

with its range of outcomes, is fixed before the test is made, so the probabilities of its outcomes cannot depend on which test is chosen later. A later choice of test is a new stance, which follows the finding (O → T) and cannot reach back. Retrocausal signaling would require reversing the cycle O → T → I → O, which Axiom II forbids. (The cycle is the T → I → O → T of Chapter 4, entered at O.)

> **Purification and the Root Tautology.**
>
> Chapter 6 of this volume reads the GNS construction as the Root Tautology in algebraic form. The purification postulate is a closely related structural fact: every state of a matrix algebra is the restriction of a pure vector state of a larger system. In finite dimension the GNS theorem and the purification postulate are two faces of one construction, as the paragraph above shows: both say that comparison is constitutive, that every partial description can be embedded in a complete one. CDP elevate this structural fact to a postulate; here it is a theorem once Chapter 6 has given the Hilbert space and the tensor product.

IV. Masanes and Müller

The Masanes-Müller program (2011) derives quantum theory from five physical requirements within the GPT framework, eliminating Hardy's simplicity axiom: the most *ad hoc* of Hardy's five. Their innovation is that, from the remaining requirements, they show that the state space of a generalized bit, a system with two distinguishable states, is a ball of dimension three, which fixes $K = N^2$ without a selection principle.

Their five requirements are: finiteness (a generalized bit is described by finitely many outcome probabilities); local tomography; equivalence of subspaces (systems that carry the same amount of information have equivalent state spaces); symmetry (every pure state is carried to every other by a continuous reversible transformation); and that all measurements are allowed.

The key structural result is the explanation of why the Bloch ball is three-dimensional. Masanes, Müller, Augusiak and Pérez-García (2014) show that, with local tomography and continuous reversible dynamics, two generalized bits whose state spaces are balls of dimension d have entangled states, and so can interact, only when $d = 3$, the quantum case, in which the reversible transformations of one bit are the rotations that $SU(2)$ induces on its ball. The three-dimensionality of the Bloch ball is thus derived from information-processing constraints.

The IM framework provides an independent route to the same result. The $\mathbb{H}$ sector of A_F is four-dimensional over $\mathbb{R}$, its unit group is $SU(2)$, and its ray space is S^2, the surface of a three-dimensional ball. The three-dimensionality is not derived from information-processing constraints but from the algebraic structure of comparison itself: the Frobenius theorem allows exactly one four-dimensional associative real division algebra ($\mathbb{H}$), and the Three-Carrier Theorem makes it the immanent carrier.

The two derivations reach the same sphere from disjoint premises. Masanes and Müller start from operational requirements about information; IM starts from the structure of comparison. The result does not depend on the specific foundational assumptions of either but follows from structural constraints that both programs, each in their own language, are expressing.

What Masanes and Müller assume that IM derives includes the GPT framework itself, the existence of distinguishable states, tomographic locality, and continuous reversibility. The existence of a generalized bit, a system with two distinguishable states, is an axiom in their program. In IM, the existence of the qubit is a derived consequence: the Frobenius theorem and the Three-Carrier Theorem guarantee that the $\mathbb{H}$ sector exists and acts on $\mathbb{C}^2$. The qubit is not postulated; it is the mandatory content of the immanent modality.

V. QBism

QBism stands apart from the three programs discussed above. In its main statement it is not a derivation program but an interpretive stance. It takes the full Hilbert space formalism (states, observables, the Born rule, unitary evolution) as given and reinterprets the meaning of probability. In QBism, quantum states are not objective features of physical systems but represent an agent's personal beliefs about the outcomes of future measurements (Fuchs, Mermin and Schack, 2014). The Born rule is not an objective law but a coherence constraint on beliefs, analogous to Dutch-book arguments in Bayesian probability theory.

Its technical program rewrites the Born rule through a SIC-POVM as a modification of the law of total probability, and asks which further postulates on that form would recover the formalism (Fuchs and Schack, 2013); it does not reach the full formalism, so the table records the formalism as assumed.

The IM framework and QBism share one important commitment: the centrality of the epistemic process. Both insist that the relationship between an observer and a system, what IM calls the comparison relation, is constitutive of physical description, not an afterthought appended to an objective dynamics. The difference is structural. QBism locates the epistemic process in the psychology of agents; IM locates it in the structure of comparison itself. QBism's epistemic process is subjective and agent-dependent; the act of comparison is structural and agent-independent. QBism reinterprets probability; IM derives it from the ICT. QBism accepts the formalism; IM derives it from axioms.

QBism and the measurement problem.

QBism claims to dissolve the measurement problem by denying that "collapse" is a physical process: it is an update of the agent's beliefs. IM also dissolves the measurement problem (Chapter 9), but through a different structural argument: the pair of the subjective and the objective, from the six intrinsics, provides a structural resolution in which measurement is a specific comparison, not a collapse of an objective state. The IM dissolution goes one step further: it grounds that pair in the Root Tautology, where QBism takes the reading of states as beliefs as its starting point. The two accounts are compatible in spirit and differ in what they take as given.

VI. Constructor Theory of Information

Constructor theory (Deutsch, 2013; its theory of information, Deutsch and Marletto, 2015) proposes to reformulate the foundations of physics in terms of which transformations (tasks) are possible and which are impossible. It is a meta-theory: it makes statements about subsidiary theories (including QM) rather than being a theory within the standard framework.

The constructor theory of information defines an information medium (a system whose states can carry distinguishable messages) and a superinformation medium (an information medium with two information variables whose union is not an information variable; no-cloning and the other features of quantum information are then derived). Quantum theory is characterized as a specific superinformation theory satisfying locality constraints. The no-cloning theorem emerges as a constructor-theoretic law, and a version of the Born rule is argued to follow from decision-theoretic considerations (Marletto, 2016).

Constructor theory and IM share a structural commitment: both seek principles more fundamental than quantum mechanics itself. But they differ in kind. Constructor theory is about what is possible and what is impossible; IM is about what is *necessary*: what must be the case given the three axioms. Constructor theory is a meta-theory that makes claims about the logical structure of physical laws; IM is a theory that derives specific mathematical structures (the comparison algebra, the spectral triple) from which specific physical laws follow. Constructor theory presupposes the notion of physical transformation; IM's axioms are more primitive, grounding the concept of transformation itself in the structure of comparison.

The two programs are not rivals. The two differ in what they take as given and in what they reach: constructor theory takes transformation and the possible/impossible distinction as primitive; IM grounds transformation in comparison and reaches the specific algebra.

VII. Chamseddine and Connes

One program does reach the specific algebra. In the non-commutative geometry of Connes, the Standard Model coupled to gravity is a spectral triple on the product of spacetime with a finite space, and its finite algebra is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. Chamseddine and Connes (2007, 2008) showed that this algebra need not be put in by hand. They classified the finite real spectral triples of KO-dimension 6 whose representations are irreducible, and found that the complexified algebra is $M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$. With a symplectic structure on the Hilbert space, which they call quaternion linearity, the algebra is $M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})$; a grading that acts non-trivially requires $a \geq 2$, and they take $a = 2$, the simplest case, which gives sixteen fermions per generation. The grading reduces $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ to $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$, and the order-one condition on the Dirac operator, with a non-

zero Majorana coupling for the right-handed neutrino, leaves $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ as the unique maximal subalgebra. The number of generations is put in.

Their hypotheses are conditions on a spectral triple, motivated by physics and by irreducibility. In *Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 10, I set the hypotheses of Chamseddine and Connes beside the conditions that I use in this book, which come from comparison:

Chamseddine and Connes	**The structure of comparison**
KO-dimension 6, from the removal of fermion doubling	KO-dimension 6, from the two terms of a state and the content/context pair (Chapter 8)
Irreducibility; quaternion linearity	Three carriers of distinct character, from Axiom III (Chapter 5)
A non-trivial grading, $a \geq 2$; the simplest case $a = 2$	Re-projection: $n = 3$; confirmed by the closure with $N = 4$ (Chapter 5)
The grading, the order-one condition, a Majorana coupling	The conjugacy condition: the omniscient and transcendent carriers of complex type (Chapter 5)
Result: $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	Result: $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

The two lists are not translations of one another. Chamseddine and Connes pass through a larger algebra and cut it down with the grading and the order-one condition; the derivation of Chapter 5 assembles the algebra from its carriers and fixes the one remaining integer by re-projection. Where the lists overlap, at KO-dimension 6, they reach the same value for independent reasons. Their choice of the simplest case, $a = 2$, also meets the closure of Chapter 5, through identifications that read the lepton as a fourth color; that reading belongs to the particle content, and *The Connes Correspondence*, Chapter 17, Section III, takes it up. The order-

one condition itself has a reading in comparison: the real structure makes the two terms of a state independent, and the order-one condition says that comparison respects that independence.

One item of their list is not shared. In their construction the Majorana coupling selects the algebra; the derivation from comparison reaches the same algebra without it. What follows for the neutrinos when the coupling is not used belongs to the particle content, and *The Branches of the Kinds*, Chapter 1, takes it up.

The relation is not rivalry. Their result shows which conditions on a spectral triple select the algebra of the Standard Model; the structure of comparison gives the reasons for the KO-dimension, for the three carriers, for their types, and for the rank three, by a route that never mentions particles.

VIII. The Common Gap

Across the five reconstruction and interpretive programs (Hardy, CDP, Masanes and Müller, QBism and constructor theory), a common gap emerges. It has two aspects; the route of non-commutative geometry of Section VII shares the first and not the second.

The first aspect is the probabilistic-operational framework. All three probabilistic programs (GPT and OPT: Hardy, CDP, and Masanes-Müller) take as primitive the notions of systems, states, preparations, measurements, and probabilities. The framework is the stage on which the derivation unfolds, and it is not itself derived. QBism goes further: it takes the entire Hilbert space formalism as the stage and offers an interpretation of it. Constructor theory operates at an even more abstract level, taking the notion of physical transformation as primitive.

IM claims to derive the stage itself. The Root Tautology grounds the concept of interaction. The six intrinsics ground the concepts of system and measurement: the apparatus is the perspective from which a comparison is made (subjective), and the system is what is compared (objective);

the property probed is the content, and the apparatus's setting and reference scale are the context; the result is a verdict of sameness or difference. The three axioms ground the distinction between omniscient, immanent, and transcendent: the structural prerequisites of any physics that distinguishes structure from act from stance and possibility. The probabilistic-operational framework is not assumed; it is a consequence of the Hilbert space that the GNS construction builds (Chapter 6) and of the Born rule (Chapter 7).

The second aspect is specificity. All the reconstruction programs derive at most "some quantum theory": complex Hilbert spaces of unspecified dimension, with unspecified gauge groups, hosting unspecified particles. They derive the *form* of quantum mechanics but not its *content*. They show that physics must be quantum but not that it must be *this* quantum: the quantum of the Standard Model, with three generations of fermions, with $U(1) \times SU(2) \times SU(3)$ as the gauge group, with the specific particle masses observed in experiment.

Only the route of non-commutative geometry of Section VII reaches the specific algebra, and it does so from conditions taken as given. Reaching the specific algebra is the advantage that IM claims. The derivation from the structure of comparison through the ICT and the three axioms through the comparison algebra A_F through the spectral triple through the spectral action yields not only "a quantum theory" but the specific algebra and gauge group of the quantum theory that describes the observed world; and it stops, as Chapter 2 requires it to say, at the masses and mixings. The gauge group $U(1) \times SU(2) \times SU(3)$ is the unitary group of A_F, reduced by the unimodularity condition, an input (Chapter 5, Section VIII). The wider program takes up the particle spectrum and the measured constants.

Three further points of comparison

The additivity premise of the Born rule. Hardy, CDP, and Masanes-Müller all assume a probabilistic framework in which additivity is built in from the start. The IM derivation supplies both premises of Gleason's theorem (Chapter 7): that a finding has one weight in every measurement that contains it, which is the law of symmetry applied to findings, stated there as an identification; and that the weights of the exclusive outcomes of one measurement add, which follows from the content/context partition, and again from the count of repeated outcomes.

The Tsirelson bound. The bound $2\sqrt{2}$ holds in any algebra of observables in which the two parties' observables commute with one another, and it is reached only when each party's own two observables fail to commute and the joint state is entangled; so every program that reconstructs Hilbert space obtains it, and Pawłowski and colleagues (2009) derive it from a principle about information alone. What the IM supplies is the reason the observables do not all commute: the algebra has two non-commutative carriers, $M_3(\mathbb{C})$, which the continuously asymmetric mode requires, since an observable that commutes with the Dirac operator without being central exists only in a non-commutative algebra, and $\mathbb{H}$, which the carrier condition requires; and its one commutative carrier gives back the classical bound 2 (Chapter 8, Section III).

The spin-statistics connection. The standard proof relies on relativistic quantum field theory (Pauli, 1940), and nonrelativistic routes have also been given (Berry and Robbins, 1997). The IM route is its own: the principle of identity, applied to content, which is interexclusionary, and to context, which is intercompositional, makes the states of many instances of content antisymmetric and those of context symmetric, and the fields take their poles by position in the act of comparison (Chapter 8); the spins of the two kinds of field are a separate fact.

The octonions. A family of proposals builds the structure of the Standard Model on the octonions or on the exceptional Jordan algebra of three-by-three octonionic matrices (Dixon, 1994; Furey, 2016; Dubois-Violette and Todorov, 2019; Boyle, 2020). The structure of comparison excludes both at the first step: comparisons composed in a chain give one result however the chain is grouped, and neither has a product that comes from an associative algebra (Chapter 5, Sections I and IV).

The assumptions of the existing programs.

The existing programs assume the following, which a deeper derivation might supply. (A) A probabilistic or operational framework. (B) Systems and states as primitive notions. (C) The specific number field (complex numbers rather than reals or quaternions). (D) Composition rules for systems (the tensor product or equivalent). (E) The specific algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. (F) The connection to particle physics. The reconstruction programs derive some of (A) to (D) and assume the rest; Chamseddine and Connes derive (E) from conditions of their own. I derive (A) to (E) from the structure of comparison, through the theorem, the three axioms and the identifications I name at each step; (F) belongs to the wider program.

IX. What Each Program Contributes

The comparison should not be read as a dismissal of the other programs. Each has made a contribution that the IM program builds upon, draws on, or is enriched by.

Hardy demonstrated that quantum theory can be *derived* from operational axioms rather than only postulated, that the project of operational derivation is not quixotic but achievable. Hardy's paper opened the operational reconstruction program (Darrigol, 2015, examines its axioms). Earlier reconstructions in quantum logic (Birkhoff and von Neumann, 1936; Mackey, 1963; Piron, 1964; Ludwig, 1985) reached the lattice of sub-

spaces, and Solèr's theorem (1995) narrowed its number field to the reals, the complexes or the quaternions; Hardy's operational route fixed the complexes. The IM derivation inherits this demonstrative achievement.

CDP showed that, among their six principles, purification is the one that separates quantum theory from classical theory: with the other five it singles out quantum theory among operational-probabilistic theories. This sharpens the question that IM must answer, and Section III answers it: once Chapter 6 has given the Hilbert space and the tensor product, purification is a theorem, and the GNS construction gives it.

Masanes and Müller provided the cleanest structural explanation of the three-dimensionality of the Bloch ball. The IM derivation arrives at the same result through a different route (the Frobenius theorem rather than group-theoretic constraints on information processing), and the two arguments reach the same result from disjoint premises.

QBism insisted on the epistemic process (on the centrality of the observer's relationship to the system) at a time when this emphasis was unfashionable in physics. The act of comparison, in the IM, is more structural and less psychological than QBism's epistemic process, but the two programs share a recognition that physics cannot be understood without understanding the epistemic relation that constitutes physical description.

Constructor theory introduced the distinction between possible and impossible transformations as a foundational principle: a distinction that is close to IM's concern for structural necessity. The two programs share the ambition to identify principles more primitive than the specific laws of physics.

Chamseddine and Connes showed that the algebra of the Standard Model can be selected by conditions on a spectral triple rather than put in by hand. That result is the closest point of contact between the axiomatic study of quantum theory and particle physics, and it is the point at which the IM's derivation of the algebra and theirs meet.

X. Summary

Every existing program that attempts to derive quantum mechanics from more primitive principles assumes at least one feature that the Immanent Metaphysics derives. The three probabilistic programs (GPT and OPT: Hardy, CDP, Masanes-Müller) assume the probabilistic-operational framework; IM derives it. QBism assumes the Hilbert space formalism; IM derives it. Constructor theory assumes the notion of physical transformation; IM derives the structural basis for transformation from the comparison relation. Chamseddine and Connes derive the specific algebra from conditions on a spectral triple; the structure of comparison reaches the same algebra from conditions of its own, and the closure of Chapter 5 meets their choice of $a = 2$.

Among the reconstruction programs, none derives the specific algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and none dissolves the measurement problem by grounding the division of every comparison into a subjective and an objective side, from which the system/apparatus distinction follows. The route of non-commutative geometry reaches the algebra and connects it to particle physics, but takes its conditions as given. The contribution of the IM is to derive the algebra from the structure of comparison, and with it the reasons for the conditions it shares with that route.

The reconstruction programs do not attempt what IM claims to achieve: the derivation of the quantum formalism and the specific algebra of the actual world from the structure of comparison, through identifications each named as one. The question for the reader is whether the derivation chain is sound in structure and wider in scope than any alternative. The comparison table sets out the difference in scope; the chain itself is set out in Chapters 3 to 9, each step named as a theorem or an identification.

KEY CONCEPTS IN THIS CHAPTER

- **Comparison tables**: Two diagnostic tables assessing, for each of six features of QM (Hilbert space, Born rule, superposition, entanglement, gauge group, particle content), whether each axiomatic program assumes or derives it. IM derives the first five in this book; the reconstruction programs derive at most four.
- **Common gap of the probabilistic programs**: All three probabilistic programs (GPT and OPT: Hardy, CDP, Masanes-Müller) assume a probabilistic-operational framework and derive at most "some quantum theory" without determining the specific algebra, gauge group, or particle content.
- **Purification from the GNS construction**: In finite dimension, the GNS space of a faithful state is its purification, and any two purifications on one purifying system differ by a unitary on that system. Identified in the Immanent Metaphysics, the purifying system is the context that an incomplete comparison traced over.
- **Hardy's axioms from IM**: IM derives four of Hardy's five axioms (probabilities from the ICT, subspaces from corner algebras, tomographic locality from the complex tensor product, continuity from the transitive action of the unit groups) and makes simplicity unnecessary.
- **Specificity**: The structural contribution of IM is the derivation of the *specific* algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and its gauge group, with reasons for the conditions it shares with the route of non-commutative geometry.
- **QBism as interpretive, not derivational**: QBism assumes the Hilbert space formalism and reinterprets probability; its technical program does not reach the full formalism, and it contributes the insistence on the centrality of the epistemic process, which IM shares at a structural rather than psychological level.

- **Constructor theory as a meta-theory**: Constructor theory characterizes the logical structure of physical laws (possible vs. impossible tasks) at a meta-level; IM derives the specific algebra and the formalism. The two differ in what they take as given and in what they reach.
- **Generalized probabilistic theory (GPT)**: An abstract framework in which a system is described by a state that determines probabilities of measurement outcomes. The framework within which Hardy and Masanes-Müller work; CDP work in the more general framework of operational-probabilistic theories (OPTs).
- **Tomographic locality**: The assumption that the state of a composite system is completely determined by the statistics of local measurements on its components.
- **SIC-POVM**: A symmetric informationally complete positive operator-valued measure. The structure in terms of which QBism rewrites the Born rule.
- **The Chamseddine-Connes classification**: The selection of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from finite spectral triples of KO-dimension 6 that are irreducible, quaternion linear and non-trivially graded, in the simplest case $a = 2$, by the order-one condition with a Majorana coupling. The structure of comparison reaches the same algebra from its own conditions.

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PART V

The Passage



Chapter 12

The Quantum Passage

What It Means That Quantum Mechanics
Is a Special Case of the Structure of Comparison

ABSTRACT

This closing chapter draws together the argument of the book. Quantum mechanics is not the foundation of physics. It is a consequence of something more fundamental: the structure of comparison. The passage from the Immanent Metaphysics to quantum mechanics is a descent from the general to the specific, from the necessary to the contingent, from the structure of any possible act of knowing to the particular formalism that governs the physical world. The chapter sets out where each postulate of quantum mechanics comes from, and what the chain rests on. Three independent roads, philosophical, categorical, and algebraic, converge on the same structural content. The "weirdness" of quantum mechanics is an artifact of starting from classical concepts and being surprised when they fail. Starting from comparison, there is nothing surprising about quantum mechanics at all.

I. The Passage

The title of this book names a specific intellectual movement: the *passage* from the Immanent Metaphysics to quantum mechanics. That passage has now been traced across the preceding chapters. It begins with the

Root Tautology: the observation that interaction, comparison, and relation are the same concept, and that this identification cannot be denied without self-contradiction. It proceeds through the six intrinsics, the Incommensuration Theorem, the three axioms, the comparison algebra, the GNS construction, the Born rule, and the specific phenomena of superposition, entanglement, spin, interference, and measurement. It arrives at the full structure of quantum mechanics, with its specific algebra and its specific gauge group.

The direction of this passage is the opposite of the direction in which physics is usually taught. In the standard presentation, quantum mechanics is the foundation. Classical mechanics is an approximation. The world is fundamentally quantum, and our classical intuitions are an imperfect reflection of a deeper quantum reality. The passage described in this book reverses this picture. Quantum mechanics is not the foundation. It is a consequence. The foundation is the structure of comparison: a structure that is more general than any physical theory, because it is the structure of any possible act of knowing, any possible interaction, any possible relation.

The passage is a *descent*, not an ascent. It moves from the general to the specific, from the necessary to the contingent. The Root Tautology is necessary in the strongest possible sense: it cannot be denied without instantiating the structure it denies. The Incommensuration Theorem is a structural fact about the nature of knowing. The three axioms state the relations the modalities must bear. The comparison algebra follows from the theorem and the axioms through the identifications of Chapter 5, each stated as one. The Hilbert space follows from the comparison algebra by the GNS construction, in the faithful complex context of evaluation that Chapter 5 states as an identification. At each step, the derivation narrows the possibilities until only the actual algebra and its gauge group remain, and what is left open is named: the entries of the finite Dirac operator. The physics is not *chosen* from a landscape of options. It is the unique structure that survives the constraints.

II. Where Each Postulate Comes From

The six postulates of Chapter 1 can now be set beside their sources.

Postulate	Where it comes from	Chapter
1. Hilbert space	The GNS representation of the comparison algebra, complex because of the faithful complex context of evaluation, read from the conjugacy clause of Axiom I; on the arena, the GNS space of its functions, $L^2(M)$, with the spinor module of its Dirac operator	5, 6
2. Observables	The self-adjoint elements of the complex algebra that A_F generates on that space, $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$; the involution that reverses a comparison becomes the adjoint	6
3. Unitary evolution	Evolution between comparisons as a continuous group of automorphisms of the algebra, unitary by the theorems of Wigner, Kadison and Stone; its generator from the Dirac operator through the spectral action	6
4. Tensor products	Two comparisons performed together without disturbing each other, an identification; the tensor product of their modules and of their algebras, by a theorem, and the GNS construction on it	6
5. Measurement outcomes and the state after an outcome	The spectral theorem for the outcomes; repeatability and the unchanged context of a completed comparison for the update; for measurements that are not ideal, an ideal comparison of a probe joined to the system	9
6. The Born rule	One weight for one finding in every measurement that contains it (the law of symmetry applied to findings); additivity over the exclusive outcomes of one measurement, from the content/context	7

Postulate	Where it comes from	Chapter
	partition and from the count of repeated outcomes; the theorems of Gleason and Busch, on the finite sectors and on the full space	

The phenomena that the postulates produce come from the same sources.

Phenomenon	Where it comes from	Chapter
Superposition	The vector-space structure of the Hilbert space; color and charge superselected by the law of symmetry	8
Entanglement	The tensor product of algebras	8
Interference	The linear structure of H_F; phases from the complex type of the carriers	8
Bell nonlocality and the Tsirelson bound	Non-commuting observables on both sides, from the two non-commutative carriers, and an entangled state; the bound by Cirel'son's theorem, reached in the smallest case on $\mathbb{C}^2 \otimes \mathbb{C}^2$, the module of $\mathbb{H} \otimes \mathbb{H}$	8
Spin and weak isospin	The unit quaternions, acting once in each mode of the ICT	6, 8
The Dirac equation and antimatter	The fermionic term of the spectral action; the real structure J, which carries the conjugacy of Axiom I to the states	6
Fermi and Bose statistics	The principle of identity on each pole, content interexclusionary and context intercompositional; the fields placed on the poles by position in the act of comparison	8
The uncertainty principle	The conjugation of Axiom I; the commutator $[x, p] = ih$, with momentum the generator of translations, and the Robertson inequality	4, 6

The phenomena of measurement and of knowing come from the same sources.

Phenomenon	Where it comes from	Chapter
No observable of time	Evolution time is transcendent, the parameter of the formalism and not a content	7
Kochen-Specker	Gleason's theorem, returning the residual of the Incommensuration Theorem in dimension three or more	7
No-cloning	The linearity of the derived evolution on the Hilbert space	10
The measurement problem dissolved	The pair of the subjective and the objective and the directed cycle of Axiom II; the residual, which remains (Measurement, Proposition 8.2)	9
Wigner's friend	Different comparisons with different subjective sides, each with one outcome; the shared record, held by those who have the outcome	9

Behind them lie the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (Chapter 5), its gauge group $U(1) \times SU(2) \times SU(3)$, obtained from its unitary group by the unimodularity condition (Chapters 5 and 6), and its internal space $\mathbb{C}P^2 \times S^2$, whose rotations are its inner symmetries (Chapter 5). The spacetime in which the formalism acts is the subject of *The Gravitational Passage* (see Chapter 6, Section V, of this book).

Figures 12.1 and 12.2 draw the whole chain. Each box is one step; an arrow runs from what a step rests on to the step. Height on the page is logical depth: two boxes stand on one row only when neither rests on the other. The outline of a box says what kind of step it is.

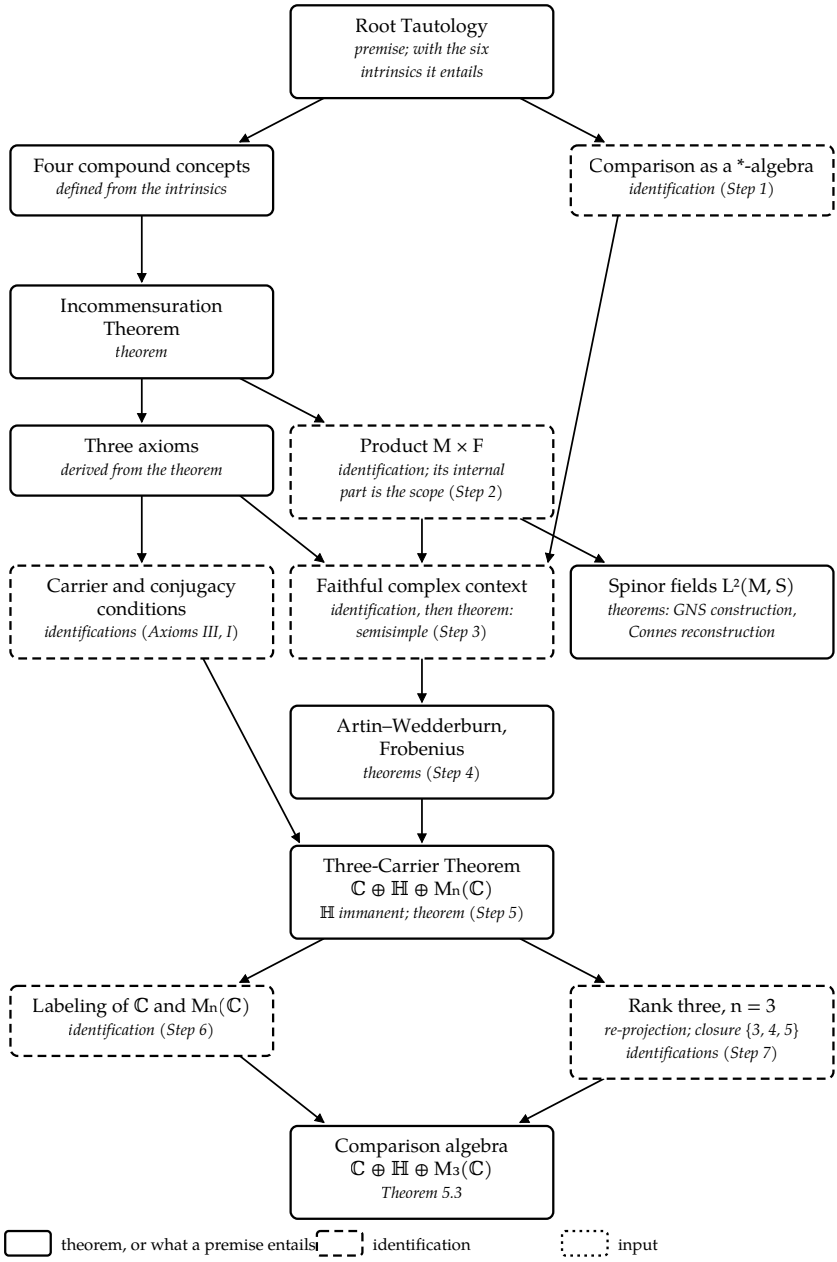

Root Tautology
premise; with the six intrinsics it entails
Four compound concepts
defined from the intrinsics
Comparison as a *-algebra
identification (Step 1)
Incommensuration Theorem
theorem
Three axioms
derived from the theorem
Product M × F
identification; its internal part is the scope (Step 2)
Carrier and conjugacy conditions
identifications (Axioms III, I)
Faithful complex context
identification, then theorem: semisimple (Step 3)
Spinor fields L²(M, S)
theorems: GNS construction, Connes reconstruction
Artin–Wedderburn, Frobenius
theorems (Step 4)
Three-Carrier Theorem
ℂ ⊕ ℍ ⊕ Mn(ℂ)
ℍ immanent; theorem (Step 5)
Labeling of ℂ and Mn(ℂ)
identification (Step 6)
Rank three, n = 3
re-projection; closure {3, 4, 5}
identifications (Step 7)
Comparison algebra
ℂ ⊕ ℍ ⊕ M3(ℂ)
Theorem 5.3
theorem, or what a premise entails
identification
input

Figure 12.1. From comparison to the algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (Chapters 3, 4 and 5).

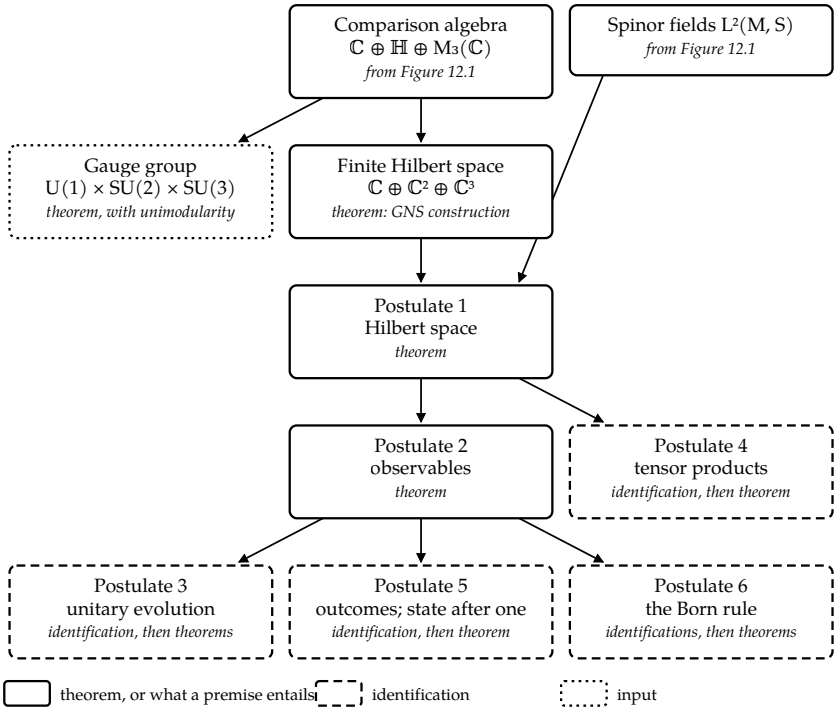


Figure 12.2. From the algebra to the six postulates (Chapters 6, 7 and 9).

What the chain rests on

The chain rests on four things.

The Root Tautology, the Incommensuration Theorem and the three axioms derived from it. The Root Tautology is the starting point, the one premise of the whole ground. Chapters 3 and 4 establish the rest: Chapter 3 grounds the six intrinsics in the Root Tautology, a tautology that cannot be denied without self-contradiction, and Chapter 4 proves the theorem from them and derives the axioms.

The stated identifications. Each carries the observation that bears on it: that every finding of physics is an act of comparison (Chapter 3); the encoding of comparison as a *-algebra; the faithful complex context of evaluation, which reads the conjugacy clause of Axiom I (Chapter 5, Section III); the carrier condition (Axiom III); the conjugacy condition (Axiom I); the labeling in the focus of the two laws; re-projection; the closure, which confirms it; the linking of every two carriers, which carries their inseparability to the representation (Chapter 6, Section III); evolution between comparisons as a continuous group of automorphisms of the algebra (Chapter 6, Postulate 3); two comparisons performed together without disturbing each other, which gives the tensor product (Chapter 6, Postulate 4); one weight for one finding in every measurement that contains it, which is the law of symmetry applied to findings, and the stable weight of a repeated comparison (Chapter 7); the reading of the real structure and the grading that fixes KO-dimension 6; and the placing of the fields on the poles of content and context by position in the act of comparison, which with the principle of identity gives the two statistics (Chapter 8).

The scope and the inputs. The algebra is that of the internal part of the product $M \times F$, a scope and not a theorem (Chapter 5, Section II). Two inputs enter: the unimodularity condition, which reduces the unitary group of A_F to $U(1) \times SU(2) \times SU(3)$ (Chapter 5, Section VIII); and the spectral action principle, which fixes which Hamiltonian generates the evolution (Chapter 6, Section VI).

The theorems of mathematics. The derivation uses the Frobenius theorem (1878), the Artin-Wedderburn theorem, the GNS construction, Connes' reconstruction theorem, the theorems of Wigner and Kadison and of Stone, the theorems of Gleason (1957) and Busch (2003), Cirel'son's theorem (1980), the Robertson inequality, and Connes' framework of spectral triples, the algebraic generalization of Riemannian geometry, whose use here is analogous to the use of differential geometry in general relativity: it is the mathematical language in which the physics is expressed, not an additional physical postulate.

Several of the postulates are reached a second way in this book, from what could be found and what is found. Possibility passes continuously between directions, and what is found is one outcome (Chapter 7, Section II); so a state can lie between directions, as a combination of several directions (Chapter 8, Section I), whose coefficients are complex by the conjugacy of the carriers (Chapter 8, Section V); and in three or more dimensions the weight that every turning of the state space keeps is the squared amplitude (Chapter 7, Section IV). Amplitudes add before weights are taken, so possibilities interfere (Chapter 8, Section V). A finding fixes the state, and nothing settles an outcome before it is found (Chapter 7, Section II; Chapter 9, Section V). Between findings the phase of a state advances at a rate set by its energy, divided by ħ, and varies across space at a rate set by its momentum, divided by ħ (Chapter 6, Section VI). Two contents of one kind cannot share a state, because exchanging them is a full turn of one, and a full turn reverses a spinor (Chapter 8, Section VI). And possibility, bound to the stance and not to a place, correlates distant findings while nothing passes between the places (Chapter 8, Section III). These routes and the algebraic road reach the same postulates.

III. The Three Roads

The structure of comparison has been stated in three languages, and the three statements form a triangle.

The philosophical road proceeds from the Root Tautology through the six intrinsics, the four compound concepts, and the examination of the ten pairings by three criteria. Exactly two pairings survive: the CA mode (continuous, asymmetric) and the SD mode (discontinuously symmetric). The axioms follow from the theorem. Its method uses no mathematics beyond the examination of ten cases, and its scope is universal: it applies to any act of knowing.

The categorical road renders the same structure in the theory of toposes. A topos is a universe of objects and maps that behaves like the universe of sets and carries its own logic; an adjunction is a pair of opposite translations between two such universes, each the best approximation to an inverse of the other. A cohesive topos is a topos joined to the universe of sets by a chain of such translations: a chain that encodes the distinction between discrete and continuous, between parts and wholes, between local and global. A cohesive topos is non-degenerate when the discrete and codiscrete objects on a set of two points differ, and it satisfies the Nullstellensatz, in Lawvere's term, when every piece of every object has a point. *Category Theory and the Immanent Metaphysics* proves that, in a non-degenerate cohesive topos that satisfies the Nullstellensatz, with a probe of an object identified as one comparison, no object with two or more points is both absolutely symmetric and absolutely continuous, and none is both absolutely asymmetric and absolutely discontinuous; the discrete objects realize the SD mode and the codiscrete objects the CA mode; and the three axioms are realized by the modalities of the topos. The step from comparison to the topos is stated there as an identification, made in the first person.

The algebraic road renders it in spectral triples, Connes' algebraic reformulation of geometry, where a comparison is carried by an algebra of observables, a Hilbert space on which they act, and a Dirac operator that measures difference. *Non-Commutative Geometry and the Immanent Metaphysics* proves the algebraic form of the theorem and gives each axiom the form of a condition on the triple; and there the structure meets the algebra that physics had found independently: $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. Chapters 5 and 6 follow that road.

The three roads assume different things. The philosophical road assumes only the six intrinsics and the criteria for conjoining their compounds; it uses no numbers, no sets and no geometry. The categorical road assumes a non-degenerately cohesive topos with the Nullstellensatz and the identification of a probe with a comparison; it uses no algebra of

operators. The algebraic road assumes the encoding of comparison as a spectral triple and its stated identifications; it uses no topos and no examination of cases, since its form of the theorem needs no criterion. Each reaches the two modes and the axioms by means the others do not use, and in each the same theorem holds in its own terms.

Only the algebraic road reaches the algebra, and there it meets physics from the other side: Chamseddine and Connes reached the same algebra from the requirements of a theory of fermions, and Connes before them from the known gauge group and particles. A road that begins from comparison and a road that begins from what was found arrive at one algebra with one set of integers, and at one gauge group, assembled from the three carriers. The explanation the convergence supports is that the structure is not an artifact of any one method: it is the structure of comparison, which any sufficiently deep investigation meets, wherever it starts.

Non-Commutative Geometry and the Immanent Metaphysics, Chapter 18, names three other roads to the same algebra: from comparison, from physics and irreducibility, and from the internal spaces of the carriers. The roads of this section are the three languages in which the structure of comparison itself is stated.

IV. The Architecture of Physics

The phrase "quantum mechanics is weird" has been repeated so often that it has acquired the force of a cultural truism. Superposition is weird. Entanglement is weird. The measurement problem is weird. The Born rule is weird. The physicist's response is to accept the weirdness and calculate. The philosopher's response is to construct interpretations that make the weirdness tolerable.

The derivation in this book gives a different response. The weirdness is not a feature of quantum mechanics. It is an artifact of the starting point. If one starts from classical concepts (definite positions, definite momenta, deterministic trajectories, local causation) and then discovers that the

world does not comply, the result is surprise. The surprise is real. But it is not a fact about the world. It is a fact about the expectations that the classical starting point generates.

If one starts instead from the structure of comparison, from the Root Tautology and the six intrinsics, there is nothing surprising about quantum mechanics at all. Superposition is a consequence of the vector space structure of the GNS Hilbert space. Entanglement is a consequence of the tensor product. Spin is a consequence of the quaternions. The Born rule is a consequence of the ICT, which makes outcomes probabilistic, and of the two premises that comparison supplies to the theorems of Gleason and Busch (Chapter 7). The contradiction of the measurement problem does not arise, because the pair of the subjective and the objective, which gives the system/apparatus distinction, is built in from the start; what remains is the residual, which is no contradiction.

Quantum mechanics is not weird. It is the only possible physics consistent with the structure of comparison. Any universe in which comparison is possible, any universe in which anything can interact with anything else, can be compared to anything else, can stand in relation to anything else, must exhibit quantum-mechanical structure in its character, a content with discrete, exact findings and a continuous formalism (*Measurement*, Proposition 9.3(a)); the specific algebra of this universe follows in addition through identifications that carry observations of this universe: by the chain this book has traced, theorems where it states theorems and identifications where it states identifications (Section II). The six postulates of Chapter 1 are not arbitrary axioms chosen to fit the data. They are structural necessities: consequences of what it means for a world to contain anything at all.

The steps that carry observations of this universe are named at each step (Section II). The faithful complex context of evaluation carries the network tests that find the complex composition of independent systems; the conjugacy condition carries the two inequivalent actions of $\mathbb{C}$ on up-type and down-type states; re-projection carries the three colors; the clos-

ure reaches the same rank three from the triangle {3, 4, 5}; and unimodularity, an input, carries the hypercharges. The five steps that carry observations are the places where the algebra of this universe meets what is found in this universe. The character of quantum mechanics, discrete and exact findings with a continuous formalism, rests on none of them. A different observation at one of those steps would change that step and the steps that rest on it, and no other; Figures 12.1 and 12.2 show which those are.

Wigner's question answered.

Wigner famously asked why mathematics is so unreasonably effective in the natural sciences. The IM derivation gives an answer: mathematics is effective not by an unexplained fit but because both mathematics and physics are consequences of the same underlying structure, the structure of comparison. Mathematics studies the structure abstractly; physics encounters it concretely. The "unreasonable" effectiveness is the expected consequence of a shared structural source. The same source sets a limit as structural as the effectiveness: a formalism reaches its findings densely and never exhaustively, since a formalism that contained its findings exactly would perfect both characters in one mode, so the mathematical description of physics approaches every finding as closely as one likes and never coincides with it (*Measurement*, Chapter 14, Section XII).

V. What the Derivation Changes

The relationship between physics and metaphysics changes. For three centuries, physics has proceeded without metaphysical foundations. The strategy has been empirical: observe, hypothesize, test, refine. The strategy has been spectacularly successful. But it has left the foundations unexplained. The six postulates of quantum mechanics are the most precisely confirmed laws of nature, yet their source has been unknown. The IM derivation offers an answer: they come from the structure of comparison, via a chain of constrained steps that can be inspected and challenged.

The relationship between the interpretations changes. The measurement problem, the inconsistency between unitary evolution and measurement collapse, is the engine that drives the interpretation industry. The measurement problem dissolves when quantum mechanics is derived from a starting point in which the inconsistency does not arise (Chapter 9). The interpretations are then unnecessary. Each is a partial view: as a description of the part it keeps it is not wrong, and as an account of the whole it suppresses a modality (Chapter 9, Section VIII). Each keeps part of the modal structure and suppresses the rest (Chapter 9): Copenhagen keeps the immanent and is silent on the transcendent; many-worlds keeps the transcendent and the omniscient and removes the immanent; Bohmian mechanics folds the immanent into the omniscient; objective collapse folds it into the transcendent; QBism denies the transcendent ontological standing; relational quantum mechanics suppresses the omniscient as shared. None captures all three simultaneously. The interpretations are partial views of a structure that can only be seen whole from the vantage point of comparison itself.

The relationship between quantum mechanics and the rest of physics changes. The reconstruction programs derive quantum mechanics in isolation: Hilbert spaces and Born rules, but no particles, no forces, no gravity, no cosmology. The IM derivation does not stop at the Hilbert space. The comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ carries the gauge group and the internal geometry of the Standard Model. Every comparison triple that satisfies the conditions of Chapter 5 has the algebra A_F, and the Standard Model's own triple is one of them (Chapter 6, Section V); its spectral action yields the Lagrangian of the Standard Model coupled to gravity. Quantum mechanics is not an isolated theory: the chain that gives its formalism also gives the algebra of the Standard Model.

VI. The Wider Program

I carry the derivation to the quantum formalism and to the algebra on which the quantum formalism is realized. The wider program takes up the particle spectrum and the measured constants. It takes the same axioms further: to the cosmological constant (*The View from Inside*); to the nature of dark matter (*The Weight of Possibility*); to quantum error correction, the limits of simulation and the resources of quantum computation (*The Quantum Bridge; Entanglement, Error Correction, and the Limits of Simulation*); and to the relation between gravity and the quantum (*Two Theories, One World*, and *The Unquantized Arena*, written for physicists). Three books take the algebra of Chapter 5 into the particle content: *The Kinds of Particle*, Chapter 7, on the three generations; *The Connes Correspondence*, Chapter 17, on the lepton as a fourth color; and *The Branches of the Kinds*, Chapter 1, on the Dirac neutrino.

The relationship between quantum mechanics and gravity is the deepest of these. The standard approach, to quantize gravity, has been pursued for nearly a century without resolution. The IM perspective inverts the question. Gravity is not a force to be quantized. Identified in the Immanent Metaphysics, gravity is the CA mode of the ICT: the continuous, asymmetric mode that encodes the large-scale geometry of spacetime. The findings of quantum mechanics, its discrete outcomes and exact charges, are of the SD mode; its formalism rests on continuous structures, and its infinite-dimensional factor comes from the spacetime mode (Chapter 4, Section III; Chapter 6, Section V). The two modes are complementary, non-reducible, and jointly exhaustive. The attempt to quantize gravity is the attempt to reduce the CA mode to the SD mode, an attempt that the ICT forbids. Quantum gravity is not a theory waiting to be found. It is a category error waiting to be dissolved.

Where this claim is argued.

The account of gravity and the quantum is argued in full in *Two Theories, One World* and *The Unquantized Arena*. I derive quantum mechanics from the IM axioms; the claim is stated here to show the direction in which the program moves.

VII. Summary

The quantum passage is the passage from comparison to physics. It is the derivation that the structure of comparison, which cannot be denied without being instantiated, contains within itself the full structure of quantum mechanics: the Hilbert space, the observables, the unitary evolution, the tensor products, the Born rule, the state after an outcome, the superposition, the entanglement, the spin, the interference, the dissolution of the measurement problem, and, beneath them, the algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and its gauge group $U(1) \times SU(2) \times SU(3)$.

The route is traced, each postulate has its source, and the structure is in place. What lies beyond this book is the wider program: the particle spectrum, the cosmological parameters, and the relationship between gravity and quantum mechanics.

I claim that quantum mechanics is not the bedrock. It is the consequence. The bedrock is the act of comparison itself: the most primitive act there is, the act that cannot be denied, the act from which everything follows.

KEY CONCEPTS IN THIS CHAPTER

- **The quantum passage**: The derivation from the Root Tautology through the six intrinsics, the ICT, the three axioms, the comparison algebra, the GNS construction, and the Born rule to the full structure of quantum mechanics. A descent from the general to the specific.
- **Sources of the postulates**: Each of the six postulates traced to its source in the comparison algebra, its representation, its spectral triple, or the pairs of intrinsics of comparison (Section II).
- **The three roads**: Three statements of the structure of comparison, philosophical (the examination of ten cases), categorical (cohesive toposes) and algebraic (spectral triples), each assuming what the others do not; the algebraic road reaches the algebra and meets physics there.
- **The architecture of physics**: The claim that quantum mechanics is not "weird" but structurally necessary: the only possible physics consistent with the structure of comparison.
- **CA and SD modes**: The two surviving modes of the ICT: continuous-asymmetric (spacetime, gravity) and discontinuously symmetric (the findings of the internal space). Complementary, non-reducible, jointly exhaustive.
- **Quantum gravity as category error**: The claim, argued in *Two Theories, One World* and *The Unquantized Arena*, that the attempt to quantize gravity is the attempt to reduce the CA mode to the SD mode, which the ICT forbids.

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Appendix A

Two Steps Shown in Full

Every Candidate for the Three Carriers,
and the Algebraic Form of the Incommensuration Theorem

ABSTRACT

Two steps of the chain carry the most weight for a reader who checks it: the Three-Carrier Theorem of Chapter 5, which fixes the form of the algebra, and the algebraic form of the Incommensuration Theorem, which Chapters 5, 7 and 8 use. This appendix sets out both in full. I set out the Three-Carrier Theorem as an enumeration. Every algebra that could carry comparison is listed, with the condition that excludes that algebra, until one form remains. I set out the algebraic form of the Incommensuration Theorem as an argument of three steps about what an observable both symmetric and continuous can report.

I. The Three-Carrier Theorem: Every Candidate

By Chapter 5, Sections III and IV, the algebra of the internal part is a direct sum of simple summands $M_n(D)$, with D one of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Each summand has one of three characters: commutative ($n = 1$ and D is $\mathbb{R}$ or $\mathbb{C}$); non-commutative division ($n = 1$ and $D = \mathbb{H}$); non-division ($n \geq 2$). The carrier condition reads Axiom III: each summand carries one modality, each of the three modalities has a summand, and no two summands have one character. The conjugacy condition reads Axiom I: the summands that carry the omniscient and the transcendent are of complex type.

The type of a summand is read from its irreducible module. The module admits either no conjugate-linear map that commutes with the summand, or one whose square is $+1$, or one whose square is -1; the summand is then of complex, real or quaternionic type. $\mathbb{C}$ acting on $\mathbb{C}$, and $M_n(\mathbb{C})$ acting on $\mathbb{C}^n$, are of complex type. $\mathbb{R}$ and $M_n(\mathbb{R})$ are of real type: complex conjugation on $\mathbb{C}^n$ commutes with them and squares to $+1$. $\mathbb{H}$ and $M_n(\mathbb{H})$ are of quaternionic type: on the module $\mathbb{C}^2$ of $\mathbb{H}$, the map that sends (z, w) to $(-\bar{w}, \bar{z})$ commutes with every quaternion and squares to -1.

The candidates fall into the classes below, each class with the condition that excludes that class.

Candidate	Example	Excluded by
One or two summands	$M_3(\mathbb{C})$; $\mathbb{C} \oplus M_3(\mathbb{C})$	Carrier condition: some modality has no summand
Four or more summands	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	Carrier condition: with three characters, two summands share one
Three summands, two of one character	$\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$; $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$	Carrier condition: two summands of one character
Commutative summand $\mathbb{R}$	$\mathbb{R} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	Conjugacy condition: $\mathbb{R}$ is of real type
Non-division summand over $\mathbb{R}$	$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{R})$	Conjugacy condition: $M_n(\mathbb{R})$ is of real type
Non-division summand over $\mathbb{H}$	$\mathbb{C} \oplus \mathbb{H} \oplus M_2(\mathbb{H})$	Conjugacy condition: $M_n(\mathbb{H})$ is of quaternionic type
$\mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C})$, $n \geq 2$	**$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$**	**Remains**

The classes cover every candidate. A candidate has some number of summands; with three, the carrier condition makes their characters one of each, so it has one commutative summand, $\mathbb{R}$ or $\mathbb{C}$, the summand $\mathbb{H}$, and one non-division summand over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the remaining form, $\mathbb{H}$ is of quaternionic type, so the conjugacy condition gives $\mathbb{H}$ neither the omniscient nor the transcendent, and therefore $\mathbb{H}$ carries the immanent. The algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$, excluded in the third row, is the one through which the route of Chamseddine and Connes passes before the order-one condition cuts it down (Chapter 11, Section VII). The size n of the non-division summand is the one integer left, and Chapter 5, Section VII fixes it at three.

II. The Algebraic Form of the Incommensuration Theorem

Take a finite spectral triple: an algebra A, a Hilbert space on which it acts, and a Dirac operator D. Two pure states φ and ψ are at the distance

$$d(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a \text{ self-adjoint}, \|[D, a]\| \leq 1 \},$$

which is finite or infinite (Connes, 1994). Two pure states are joined by a step of inner symmetry when one is the other moved by a unitary of the algebra, $\psi(a) = \varphi(u^* a u)$. A class of comparison is a set of pure states joined to one another by chains of steps, each step of finite distance or of inner symmetry. An observable is symmetric when it is central, unchanged by every inner symmetry, and continuous when it commutes with D.

A continuous observable does not change across a step of finite distance. Let a commute with D. Then every real multiple λa also commutes with D, so $\|[D, \lambda a]\| = 0 \leq 1$ for every λ. If $\varphi(a)$ and $\psi(a)$ differed, $|\varphi(\lambda a) - \psi(\lambda a)|$ would grow without bound as λ grows, and the distance between φ and ψ would be infinite. So two states at finite distance give a continuous observable the same value.

A symmetric observable does not change across a step of inner symmetry. Let a be central. Then u*au = a for every unitary u of the algebra, so φ(u*au) = φ(a): two states joined by an inner symmetry give a symmetric observable the same value.

An observable both symmetric and continuous is constant on each class. Along any chain of steps its value never changes, since each step is of one of the two kinds. So it takes one value on each class of comparison: it can report which class a state lies in, and nothing within the class.

This statement is the form Chapter 5, Section II uses. An observable that reports anything within a class must fail to be symmetric or fail to be continuous. An observable of the continuously asymmetric mode commutes with D and is not central, and a commutative algebra has no observable that is not central; so an algebra that has such observables is noncommutative. Chapter 7, Section II uses the same form for the residual: an observable that recorded every finding deterministically would be both symmetric and continuous, and so could report the class and nothing within it.

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Appendix B

What the Derivation Says Experiment Will Find

Statements of Kind, with the Measurements That Bear on Them

ABSTRACT

The derivation of this book reaches quantum mechanics with reasons attached, and several of its results are statements of kind about what experiment finds. This appendix gathers them. Each entry says what a class of measurement measures, what the derivation says it will find, and where the book derives it. Most of the entries have already been measured, and each measurement found what the entry states.

I. The Entries

Composition of independent systems. Network experiments with independent sources compare the predictions of real and complex quantum theory. The derivation says they will find the complex composition (Chapter 5, Section III; Chapter 8, Section V). Experiments in 2022 found it (Chen and colleagues; Li and colleagues).

The strength of correlations. Tests of the CHSH inequality measure a sum of four correlations between two parties. The derivation says the sum will reach $2\sqrt{2}$ and go no further, and that it will stay at 2 or below whenever either party's two observables commute (Chapter 8, Section III).

Interference among three paths. Experiments with three or more slits measure whether interference has a term that the pairs of paths do not account for (Sorkin, 1994). The derivation says that term is zero, since the Born rule weighs findings by the squares of amplitudes (Chapter 7). Three-slit experiments have found it consistent with zero (Sinha and colleagues, 2010).

What is found of color and charge. The derivation says that color is found only in singlets, that no superposition of states of different electric charge is ever found, and that weak isospin is neither confined nor superselected (Chapter 8, Section I).

The double cover. Neutron interferometry measures the phase of a spinor turned about an axis. The derivation says a turn of 360° reverses its sign and a turn of 720° restores it (Chapter 8, Section IV), as Rauch and colleagues and Werner and colleagues found in 1975.

Exclusion and addition. The derivation says that matter excludes, two fermions of one kind never sharing a state, and that the fields of interaction add, any number of bosons sharing one (Chapter 8, Section VI), as the periodic table and the laser show.

Antiparticles. The derivation says that every particle has an antiparticle, paired with it by the real structure of the triple (Chapter 6, Section VII).

The algebra of the internal space. The derivation says that no physical system carries an octonionic sector or a sector of the exceptional Jordan algebra, since neither has a product that comes from an associative algebra (Chapter 5, Section IV).

The Born rule. Repeated measurements on identically prepared systems measure the relative frequencies of outcomes. The derivation says they will settle to the weights of the Born rule, in every sector and on the full space (Chapter 7).

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Terms

Each term of the book, with the definition it carries where it does its work. The chapter and section given are where the reader first meets the term in that sense.

Additivity condition

The requirement that probabilities of mutually exclusive outcomes sum to the probability of their disjunction. Gleason's second premise, from the content/context partition and the sameness/difference pair, and from the count of repeated outcomes. (Chapter 7.)

Antimatter

The antiparticle states that the real structure J pairs with every particle state; J carries the conjugacy of Axiom I to the states (Chapter 6, Section VII).

Associativity

Comparisons composed in a chain give one result however the chain is grouped; it excludes the octonions and the exceptional Jordan algebra. (Chapter 5.)

Axiom I

The immanent is prior to the omniscient and the transcendent, which are conjugate. (Chapter 4.)

Axiom II

The directed cycle T → I → O → T, each step many to one. (Chapter 4.)

Axiom III

The three modalities are distinct, inseparable and non-interchangeable. (Chapter 4.)

Balance criterion

In each respect, content and context, sameness must appear with difference; alone it leaves the same two survivors. (Chapter 4.)

Bell nonlocality

The violation of Bell inequalities by entangled quantum systems. Derived from non-commuting observables on both sides, which both non-commutative carriers supply in the complex context of evaluation, together with an entangled state. (Chapter 8.)

Born rule

The prescription $p(k) = |\langle\psi|e_k\rangle|^2$ connecting quantum states to measurement probabilities. Postulate 6 of standard QM. (Chapter 7.)

Born rule at the seam

The Born rule carries a state of the continuous formalism to the weights of discontinuously symmetric findings; it stands at the seam between the two levels of the theory (Chapter 4, Section III).

Busch's theorem

In every dimension, two included, the Born rule is the unique probability measure on effects that is additive over the outcomes of a measurement (Busch, 2003); it carries the result into the two-dimensional sector. (Chapter 7.)

CA mode

The continuously asymmetric mode, identified with the character of spacetime. (Chapter 4.)

Canonical commutator

$[x, p] = i\hbar$, from momentum as the generator of translations and Stone's theorem (Chapter 6, Section V).

Carrier condition

Axiom III read on the summands: one modality per carrier, a carrier for each modality, no two carriers of one character. (Chapter 5.)

Carriers

The simple summands of the comparison algebra, each carrying one modality (Chapter 5, Section V).

Chamseddine-Connes classification

The selection of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from finite spectral triples of KO-dimension 6 that are irreducible, quaternion linear and non-trivially graded, in the simplest case $a = 2$, by the order-one condition with a Majorana coupling. The structure of comparison reaches the same algebra from its own conditions. (Chapter 11.)

Characters

Commutative ($\mathbb{R}$, $\mathbb{C}$), non-commutative division ($\mathbb{H}$ alone), non-division ($M_n(D)$, $n \geq 2$). (Chapter 5.)

Class of comparison

A set of pure states joined to one another by steps of finite distance or of inner symmetry (Chapter 4, Section II; Appendix A, Section II).

Closure

$c_1^2 + N^2 = H^2$; with $N = 4$ it gives $c_1 = 3$ and $H = 5$. (Chapter 5.)

Common gap of the probabilistic programs

All three probabilistic programs (GPT and OPT: Hardy, CDP, Masanes-Müller) assume a probabilistic-operational framework and derive at most "some quantum theory" without determining the specific algebra, gauge group, or particle content. (Chapter 11.)

Comparison algebra (A_F)

The algebra of the internal part of comparison, $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. (Chapter 5.)

Composite systems

Two comparisons performed together without disturbing each other (an identification); their joint algebra is the tensor product, and a state not a convex combination of product states is entangled (Postulate 4). (Chapter 6.)

Compound concepts

Continuity (s, s), symmetry (s, d), discontinuity (d, s) and asymmetry (d, d), each a pair of verdicts on content and on context. (Chapter 4.)

Conjugacy condition

The conjugacy clause of Axiom I read as complex type for the omniscient and transcendent carriers; it forces $\mathbb{H}$ as the immanent carrier and the field $\mathbb{C}$. (Chapter 5.)

Content and context

What a comparison is about, and the respect in which the two things compared are two. (Chapter 3.)

Content/context partition

The structural division of every comparison into what it is about (content) and the respect in which the things compared are two (context). One of the three intrinsic pairs. (Chapter 7.)

Context of evaluation

The complex Hilbert space on which the observables act faithfully; an identification, from which semisimplicity follows as a theorem (Chapter 5, Section III).

Criterion of derivation

Four conditions: the axioms do not presuppose the target formalism; every step follows from its predecessors; the derivation reaches the specific algebra, or states where it stops; every identification is stated as one. (Chapter 2.)

Decoherence

The loss of quantum coherence through interaction with the environment. The outward movement of the content/context boundary through entanglement with a larger context, so that the coherence terms belong to the joint system. Not the same as collapse. (Chapter 9.)

Derivation

A chain of reasoning from axioms to target formalism in which every step follows from what precedes it, with no smuggled assumptions. (Chapter 2.)

Dirac equation

The equation of motion $D_A\xi = 0$ of the fermionic term of the spectral action; in Lorentzian form, $(i\gamma^\mu D_\mu - m)\psi = 0$ (Chapter 6, Section VII).

Dirac operator

The self-adjoint operator of a spectral triple that measures difference between states; on the internal part its entries are the masses and mixings of the fermions (Chapter 5, Section II).

Directed cycle

The cycle $T \to I \to O \to T$ of the three modalities (Axiom II), entered at the omniscient as $O \to T \to I \to O$. Provides structural irreversibility for the measurement transition without an additional dynamical law. (Chapter 9.)

Entanglement

A joint state of two systems that cannot be decomposed into independent states of the parts. Derived from the tensor product, which Chapter 6, Section VI derives from the composition of two comparisons that do not disturb each other.

Focus

What a comparison is being considered as; which pair of intrinsics takes which role depends on it. (Chapter 3.)

Frame of measurement

The set of units that *The Story of the Particles* uses, the frame "in which the comparing act is light" (*The Kinds of Particle*, Chapter 4, Section III). In this frame the labeling of the carriers is a rotation of the labeling made in the focus of the two laws; in it $\mathbb{H}$ carries the transcendent, and spin is the transcendent label of a particle (Chapter 3, Section V; Chapter 8, Section IV).

Gate groups from the algebra

Single-qubit gates are the unitaries of $\mathbb{C}^2$, the unit quaternions $SU(2)$ up to a phase. The composite algebra $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$ gives only local gates and the exchange; the entangling gates are unitaries of the complex space $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose operators form $M_4(\mathbb{C})$, with gate group $U(4)$, $SU(4)$ up to a phase. These gate groups are derived from the algebra, not assumed from engineering convention. (Chapter 10.)

General measurement

An ideal comparison of a probe joined to the system; tracing over the probe gives $\varphi_k(a) = \varphi(K_k{}^* a K_k) \,/\, \varphi(K_k{}^* K_k)$, with the Lüders rule as its ideal case (Chapter 9, Section V).

Generalized probabilistic theory (GPT)

An abstract framework in which a system is described by a state that determines probabilities of measurement outcomes. The framework within which Hardy and Masanes-Müller work; CDP work in the more general framework of operational-probabilistic theories (OPTs). (Chapter 11.)

Gleason's theorem

The mathematical result (1957) that, on a separable Hilbert space of dimension ≥ 3, every assignment of probabilities to subspaces that depends only on the subspace and is additive on orthogonal subspaces has the form $\mathrm{tr}(\rho P)$; for a pure state, the Born rule. (Chapter 1.)

GNS construction

The Gel'fand-Naimark-Segal theorem: given a C*-algebra A and a state φ, the construction yields a Hilbert space H_{φ}, a representation π_{φ}, and a cyclic vector Ω_{φ}. Read as the Root Tautology in the three-column dictionary. Applied to the functions on spacetime, it gives $L^2(M)$, and with the spinor module of the Dirac operator, $L^2(M, S)$ (Chapter 6, Section V).

Grading γ

The self-adjoint operator of square one that divides the Hilbert space of the triple into left-handed and right-handed states (Chapter 6, Section III).

Heisenberg cut

The boundary between system and apparatus in the standard formulation. Arbitrary in the standard framework; in the IM framework, present in every comparison and set by the comparison performed. (Chapter 9.)

$H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3$

The finite Hilbert space obtained by applying the GNS construction sector-by-sector to A_F. Total dimension 6, the smallest faithful representation. (Chapter 6.)

Hilbert space

A complete inner-product space over the complex numbers. The arena in which quantum states live, and the content of Postulate 1. (Chapter 1.)

Identical particles

Particles that are in principle indistinguishable. A consequence of the finite sectoral structure of A_F: excitations in the same sector carry identical quantum numbers. (Chapter 8.)

Identification

A stated step from the structure of comparison to a physical quantity or mathematical condition, with the observation that bears on it; a kind of step, not a grade. (Chapter 2.)

Immanent Metaphysics (IM)

The formal account of the structure of comparison from which the derivation proceeds (Chapter 1, Section IV; Chapter 3).

Immanent modality

The role of the act itself, here and now; in a single comparison, the act of comparing. (Chapter 3.)

Immanent transition

The passage from possible to actual in the directed cycle. The structural analogue of the "collapse" in the standard formulation, but not a physical process. (Chapter 9.)

Incommensuration Theorem (ICT)

Symmetry and continuity cannot both be perfected in one comparison, nor asymmetry and discontinuity; a comparison perfected at its foundation is continuously asymmetric or discontinuously symmetric. (Chapter 4.)

Inseparability

The structural property that wherever one member of a pair (e.g. sameness) appears, the other (difference) is necessarily implicit. (Chapter 3.)

Interference

The constructive and destructive combination of probability amplitudes. Derived from the linear structure of the Hilbert space: amplitudes add before probabilities are taken. Its phases are complex because the context of evaluation is complex (Chapter 5, Section III).

Intrinsics of comparison

The six concepts (sameness, difference, content, context, subjective, objective) that are both necessary and sufficient for any act of comparison. (Chapter 3.)

KO-dimension 6

The signs of the real structure J and the grading γ, fixed by the two terms of a state and by content and context. (Chapter 8.)

Kochen-Specker theorem

In dimension three or more, no assignment of the values 0 and 1 to every projection meets Gleason's two premises; the residual of the Incommensuration Theorem, returned as a theorem of the Hilbert space (Chapter 7, Section III).

Labeling

$\mathbb{C}$ omniscient, $\mathbb{H}$ immanent, $M_n(\mathbb{C})$ transcendent, in the focus of the two laws; exchanging two labels would reverse Axiom II. (Chapter 5.)

Law of continuity

Continuity applies to all that is of context (*Comparison; The Necessary Basis*). (Chapter 3.)

Law of symmetry

Symmetry applies to all that is of content; what is found as content is found as a sameness of content across a difference of context (*Comparison; The Necessary Basis*). (Chapter 3.)

M × F

The two modes composed side by side, the conjunction mediated by scope. (Chapter 4.)

Measurement problem

The inconsistency between unitary evolution (Postulate 3) and collapse (Postulates 5 and 6) in the standard formulation of quantum mechanics. (Chapter 9.)

Measurement update

The replacement of the state by the eigenstate (or normalized projection) belonging to the outcome found. The content of Postulate 5. (Chapter 1.)

Modal map of the interpretations

The arrangement of interpretations by which modality they suppress: Copenhagen (silent on the transcendent, and so unable to place the cut), many-worlds (suppresses immanent), Bohmian (reduces immanent to omniscient), QBism (denies ontological transcendent), objective collapse (folds immanent into transcendent), relational quantum mechanics (suppresses the omniscient as shared). Each interpretation's failure mode corresponds to its suppressed modality. (Chapter 9.)

Modalities

The three roles in which every comparison stands: the omniscient, the immanent and the transcendent (Chapter 3, Section IV).

Modular flow

The one-parameter group of automorphisms that a faithful state determines by itself (Tomita and Takesaki); for an equilibrium state it is the evolution generated by the Hamiltonian; read in the IM as the time an act of comparison carries. (Chapter 6.)

No-cloning

The impossibility of copying a state that is not one of a set of perfectly distinguishable states. It follows from the linearity of the derived evolution on H_F; the outcomes of one setting can be copied. (Chapter 10.)

Non-commutativity

The property that $ab \neq ba$ for certain elements of A_F. It has two sources: the Incommensuration Theorem, since an observable of the continuously asymmetric mode commutes with the Dirac operator without being central, which only a non-commutative algebra allows ($M_3(\mathbb{C})$); and the carrier condition, which requires $\mathbb{H}$. On both sides of an entangled state, it is the condition of Bell violation. (Chapter 8.)

Non-contextuality of the weights

The weight of a finding depends only on the subspace that represents it, whatever complete measurement contains it; Gleason's first premise, from the law of symmetry (an identification). (Chapter 7.)

Non-interchangeability

The property that the three pairs play structurally different roles and cannot be substituted for one another. (Chapter 3.)

Omniscient modality

The role of what is laid out as structure; in a single comparison, the thing compared. (Chapter 3.)

Probability from repetition

The same comparison, performed on the same state in the same context, is the same comparison, so its outcomes recur with one stable relative frequency, the probability (an identification). (Chapter 7.)

Purification

In finite dimension, the GNS space of a faithful state is its purification, and any two purifications on one purifying system differ by a unitary on that system. Identified in the Immanent Metaphysics, the purifying system is the context that an incomplete comparison traced over. (Chapter 11.)

Quantum gravity as category error

The claim, argued in *Two Theories, One World* and *The Unquantized Arena*, that the attempt to quantize gravity is the attempt to reduce the CA mode to the SD mode, which the ICT forbids. (Chapter 12.)

Quantum passage

The derivation from the Root Tautology through the six intrinsics, the ICT, the three axioms, the comparison algebra, the GNS construction, and the Born rule to the full structure of quantum mechanics. A descent from the general to the specific. (Chapter 12.)

Qubit

The qubit is the module $\mathbb{C}^2$ of the quaternionic carrier of A_F, its smallest faithful complex module, with the Bloch sphere S^2 as its ray space. The spin of an electron carries this module directly; other experimental qubits carry it as their state space. (Chapter 10.)

Ray space

The space of complex lines in a module; for $\mathbb{H}$ it is the sphere S^2, while its states as a real algebra reduce to one. (Chapter 5.)

Re-projection

The content repeats the structure of comparison: three colors from the three positions of comparison, the modalities. (Chapter 5.)

Real structure J

The antilinear operator of the comparison triple that carries each particle state to its antiparticle state; it exchanges the two terms of a state (Chapter 8, Section VI).

Reformulation

A rearrangement of the postulates of an existing theory, which clarifies which postulates are independent without deriving the theory. (Chapter 2.)

Residual

What every lawful description leaves unfixed; for quantum mechanics, the definite finding, which the continuous formalism does not contain (*Measurement*, Proposition 8.2). (Chapter 9.)

Root Tautology

The equation of interaction, comparison, and relation as structurally identical concepts. A fixed point that cannot be denied without self-contradiction. (Chapter 3.)

Running example

The silver atom of Gerlach and Stern, followed through Chapters 3, 6, 7, 9 and 10.

SD mode

The discontinuously symmetric mode (symmetric, discontinuous), identified with the character of what is found of the internal space. (Chapter 4.)

Self-adjoint operator

An operator equal to its own adjoint, guaranteeing real eigenvalues. The mathematical representation of an observable, and the content of Postulate 2. (Chapter 1.)

Self-description

The axioms have the characters of the omniscient, immanent and transcendent, and each is one edge of the cycle. (Chapter 4.)

SIC-POVM

A symmetric informationally complete positive operator-valued measure. The structure in terms of which QBism rewrites the Born rule. (Chapter 11.)

Specificity

The structural contribution of IM is the derivation of the *specific* algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and its gauge group, with reasons for the conditions it shares with the route of non-commutative geometry. (Chapter 11.)

Spectral action

The action principle $S = \mathrm{Tr}\ f(D_A/\Lambda) + \frac{1}{2}\langle J\xi, D_A\xi\rangle$ that derives the Standard Model Lagrangian plus Einstein gravity from the spectral triple. It supplies the generator of evolution; the unitary form comes from evolution as a continuous group of automorphisms (Postulate 3). (Chapter 6.)

Spectral triple

The data (A, H, D): an algebra, a Hilbert space on which it acts, and a Dirac operator; with a real structure J and a grading γ it is the comparison triple of Section III. It encodes both spacetime and internal degrees of freedom. The full spectral triple is $(C^\infty(M) \otimes A_F, L^2(M, S) \otimes \mathscr{H}_F, D_M \otimes 1 + \gamma_5 \otimes D_F)$, with real structure J and grading γ. (Chapter 6.)

Spin

The intrinsic angular momentum of quantum particles. Derived from the quaternions: unit quaternions = SU(2), acting on the spinor fields of spacetime; the same group acts on the $\mathbb{H}$ factor as weak isospin. Spin one half is the fundamental representation of SU(2) on $\mathbb{C}^2$; a rotation by 360° reverses the sign of the state vector, as neutron interferometry confirms. (Chapter 8.)

State update

The projection rule of Lüders, $\varphi_k(a) = \varphi(P_k a P_k) \,/\, \varphi(P_k)$, derived from the repeatability of a comparison of the first kind and from the identification that a comparison which leaves its context unchanged leaves the relative weights of the state unchanged. (Chapter 9.)

Statistics

Fermi statistics for content, Bose statistics for context, from the principle of identity applied to the two poles, content interexclusionary and context intercompositional, with the fields placed on the poles by position in the act of comparison; Pauli's theorem is not used. (Chapter 8.)

Subjective and objective

The structural division of every comparison into the side from which it is made (the apparatus) and the side about which it is made (the system). Present in every comparison and found by perspective, it provides the system/apparatus distinction without an arbitrary Heisenberg cut. (Chapter 9.)

Superposition

The ability of a quantum state to be a complex linear combination of other states. Derived from the vector-space structure of the Hilbert space; only color singlets are found, and electric charge is superselected, by the law of symmetry. (Chapter 8.)

Tensor product

The composition rule for combining the state spaces of separate systems. Produces entanglement. The content of Postulate 4. (Chapter 1.)

Three roads

Three statements of the structure of comparison, philosophical (the examination of ten cases), categorical (cohesive toposes) and algebraic (spectral triples), each assuming what the others do not; the algebraic road reaches the algebra and meets physics there. (Chapter 12.)

Three times

Evolution time (transcendent), measurement time (immanent), geometric time (omniscient, time laid out as structure). Evolution and measurement are structurally distinct and cannot be reduced to each other. Evolution time is a parameter and not a content, so time is not an observable. (Chapter 7.)

Three-Carrier Theorem

The two conditions give $\mathbb{C} \oplus \mathbb{H} \oplus M_n(\mathbb{C})$, with $\mathbb{H}$ immanent. (Chapter 5.)

Three-column dictionary

A concordance recording each concept of the IM with what carries it in quantum mechanics, category theory and operator algebra. No functor joins the formal columns; each has its theorems in its own setting. (Chapter 6.)

Tomographic locality

The assumption that the state of a composite system is completely determined by the statistics of local measurements on its components. (Chapter 11.)

Transcendent modality

The role of the stance and the possibility from which an act proceeds; in a single comparison, the thing compared against. (Chapter 3.)

Translation

The recasting of an existing formalism in a new mathematical language, as phase-space mechanics recasts Newtonian mechanics. (Chapter 2.)

Tsirelson bound

The maximum quantum violation of the CHSH Bell inequality ($2\sqrt{2}$). Cirel'son's theorem on the Hilbert space of Chapter 6; it is reached only where both parties have non-commuting observables, the smallest case being the qubit. The bound follows from Landau's identity, because the commutator of two observables with outcomes ± 1 has norm at most two; and, in a state of $A_F \otimes A_F$, it is reached with either non-commutative carrier, $\mathbb{H}$ or $M_3(\mathbb{C})$, on each side. (Chapter 8.)

Two levels

A theory's formalism exists and its findings are real, in the exact senses of Chapter 3; the theorem applies to the findings.

Two senses of the measurement problem

The contradiction among the postulates, which the derivation dissolves, and the residual between a continuous formalism and a definite finding, which remains as no contradiction (Chapter 9, Section I).

Unimodularity

The determinant of the two-sided action over the Hilbert space of the Standard Model is one; it gives $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ and the hypercharges. (Chapter 5.)

Unitary evolution

Time evolution that preserves inner products. Deterministic, linear, continuous, and reversible. The content of Postulate 3. (Chapter 1.)

Wigner's friend

The friend's comparison and Wigner's are different comparisons, with different subjective sides, and each has one outcome; a conclusion drawn within one comparison is a premise for every comparison that has its outcome (Chapter 9, Section III).

$\mathcal{H}_{\mathbf{F}} = \mathbb{C}^{\mathbf{96}}$

The Hilbert space of the finite geometry of the Standard Model: sixteen particle and sixteen antiparticle states per generation, three generations; the gauge group and the hypercharges are computed on it. (Chapter 6.)

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