

Beyond the Quantum Passage

Work Held Outside the Derivation of Quantum Mechanics

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What This Book Holds

I. The Derivation and the Work Beyond It

This book is one of a collection of books that together give one account of physics. One book of the collection derives quantum mechanics from the structure of comparison: *The Quantum Passage; Quantum Mechanics from the Structure of Comparison*. The derivation runs through Chapters 3 to 7 and 9 of *The Quantum Passage*. The derivation starts from the ground. The ground is the Root Tautology, the six intrinsics (content and context, objective and subjective, sameness and difference) and the three modalities (the omniscient, the immanent and the transcendent). The derivation also uses the Incommensuration Theorem and three axioms. Axiom III states that the omniscient, immanent and transcendent modalities are distinct, inseparable and non-interchangeable (*The Quantum Passage*, Chapter 4, Section IV). The Incommensuration Theorem states that symmetry and continuity cannot both be perfected in any comparison.

From the three axioms *The Quantum Passage* derives the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The comparison algebra has three separate parts, joined by $\oplus$ (a direct sum). The three parts are called carriers: $\mathbb{C}$, the complex numbers; $\mathbb{H}$, the quaternions; and $M_3(\mathbb{C})$, the 3×3 complex matrices. A Hilbert space is the space of the states of a quantum system. *The Quantum Passage* builds the Hilbert space from the comparison algebra by the construction of the mathematicians Gelfand, Naimark and Segal (the GNS construction). *The Quantum Passage* also derives the observables (the quantities that a measurement finds), unitary evolution (the reversible change in time of a closed system) and the tensor product

(the rule that describes two systems together). *The Quantum Passage* derives the Born rule, which gives the probability of each outcome of a measurement, and the state after an outcome.

I take the derived mathematics further here, in six pieces of work, one in each of the six chapters of this book. Each piece of work starts from a result that the derivation has established, or from a result that another book of the collection establishes, and each piece then works out a consequence of the result it starts from. The derivation uses no piece of work of this book. So each piece can be read, checked and cited on its own terms. Each chapter states which book its piece of work rests on. When a chapter of this book refers to a chapter of another book, the chapter names that other book, so "Chapter 3" with no book named refers to Chapter 3 of this book.

I name each carrier by its modality in one frame, called the focus of the two laws (the two laws are symmetry and continuity). In the focus of the two laws, symmetry carries the omniscient character and continuity carries the transcendent character. *The Quantum Passage* labels the carriers in the focus of the two laws (its Chapter 5, Section VI). In the labeling of *The Quantum Passage*, $\mathbb{C}$ is the omniscient carrier, $\mathbb{H}$ the immanent carrier, and $M_3(\mathbb{C})$ the transcendent carrier. A second frame is the frame of measurement of *The Story of the Particles; Why Every Particle Is Required, and Why There Are No More*. The labeling in the frame of measurement is a rotation of the labeling of *The Quantum Passage*. The rotation moves the omniscient, immanent and transcendent labels among $\mathbb{C}$, $\mathbb{H}$ and $M_3(\mathbb{C})$. In the frame of measurement $\mathbb{H}$ carries the transcendent (*The Quantum Passage*, Chapter 8, Section IV). One structure, the comparison algebra, has two bindings of modalities to carriers: one binding in each frame. In both frames the carriers are the same three parts of the comparison algebra, and only the binding of modality to carrier differs.

II. The Six Chapters

Chapter 1 works out, step by step, the Tsirelson bound on the composite algebra $A_F \otimes A_F$. The composite algebra is the algebra that describes two separated systems together. Each of two parties measures one of the two systems, and each party chooses between two measurements. The CHSH combination is one value made of four correlations between the results of the two parties. The physicists Clauser, Horne, Shimony and Holt set out the CHSH combination in the CHSH inequality. The Tsirelson bound is the largest value of the CHSH combination that quantum mechanics allows, $2\sqrt{2}$, while any theory of local hidden variables allows a value of at most 2, which is called the classical value. *The Quantum Passage* states the Tsirelson bound for its derived Hilbert space (its Chapter 8, Section III).

Chapter 1 shows each step of the derivation of the bound. The space of the two systems together splits into nine sectors, one for each pair of carriers, with one carrier on each side. *The Quantum Passage* names the sectors in which the bound is attained. Chapter 1 shows that the sectors separate, because the operator for the CHSH combination maps each sector to itself, so the largest value over all states is the largest value found in any one sector. In 1987 Landau published an identity, now called the identity of Landau, which limits the value in every sector to at most $2\sqrt{2}$, so no sector exceeds the Tsirelson bound. Chapter 1 gives an explicit state and explicit settings of the measurements that attain the value $2\sqrt{2}$. Chapter 1 also shows that the value is at most the classical value 2 in every sector where either party has the commutative carrier $\mathbb{C}$, which is the one carrier whose elements give the same product in either order.

Chapter 2 is about what the finite algebra contributes to decoherence. The comparison algebra has finite dimension. So I also call the comparison algebra the finite algebra. Coherence is the property of a quantum state that lets the alternatives of the state interfere with one another. Decoher-

ence is the irreversible loss of the coherence of a quantum system into the surroundings of the system. The finite Dirac operator is the operator through which the carriers of the finite algebra are coupled to one another. The finite Dirac operator joins the immanent carrier $\mathbb{H}$ to the omniscient carrier $\mathbb{C}$, and never touches the transcendent carrier $M_3(\mathbb{C})$, so the transcendent carrier stays uncoupled from the other two carriers. A coupling inside the finite algebra is coherent, and so decoheres nothing, because the coupling joins finite sectors into a closed system of finite dimension, whose change in time is unitary and reversible. The environment that makes decoherence irreversible is the continuum of the spacetime factor. The continuum of the spacetime factor consists of the fields spread continuously over spacetime, which have many degrees of freedom and continuous spectra, so correlations spread into the continuum and do not return. The rate of decoherence contains a time scale, the correlation time, which belongs to the continuum, not to the finite algebra.

Chapter 3 is about one-qubit states. A qubit is a quantum system with two alternatives. In the comparison algebra, the qubit is the space $\mathbb{C}^2$ on which the immanent carrier $\mathbb{H}$ acts. A gate is one operation on one or two qubits. Clifford circuits are circuits built from Clifford gates: the Hadamard gate, the phase gate and the controlled-NOT. A classical computer can simulate Clifford circuits efficiently, as the theorem of Gottesman and Knill establishes. The states that Clifford circuits prepare are called stabilizer states. Chapter 3 compares two measures of how far a one-qubit state lies from the stabilizer states: the log-robustness of magic and the stabilizer infidelity. The log-robustness of magic is the logarithm of the least total weight with which a state can be written as a combination of stabilizer states. The stabilizer infidelity is zero exactly on the stabilizer states, and grows as the largest overlap of a state with a stabilizer state falls. Both measures can be computed from the Bloch vector of the state. The Bloch vector is the arrow $r = (x, y, z)$ that places a one-qubit state in a ball of radius one, with length one for a pure state. One qubit has six pure stabilizer states, at the six points where the three axes x, y and z cross the sur-

face of the ball. The six points are the corners of an octahedron, called the stabilizer octahedron, and the geometry of the stabilizer octahedron fixes the exact relation between the log-robustness of magic and the stabilizer infidelity.

Chapter 4 is about magic and entanglement. Universal quantum computation needs a further resource that Clifford circuits cannot supply. A state's magic is the amount of the further resource for universal quantum computation that the state carries. Entanglement is the joining of several qubits into one state that cannot be written as a separate state for each qubit. Chapter 4 shows that the entanglement made by Clifford gates spreads magic over many qubits and cannot create magic, so a state of log-robustness M needs at least 2M non-Clifford gates to prepare. The T-gate is one non-Clifford gate: a gate outside the Clifford set. Chapter 4 also shows that the count which magic constrains is the count of non-Clifford gates, not the count of the qubits that those gates act on, because swaps of qubits, which are Clifford gates, can route every non-Clifford gate through one qubit.

Chapter 5 is about the integer $c_1 = 3$, the rank of the transcendent carrier $M_3(\mathbb{C})$, which is the number of independent directions in the space $\mathbb{C}^3$ on which the transcendent carrier acts. Chapter 5 shows that the three units of c_1, one unit for each direction, weigh the same, because the symmetries of the transcendent carrier carry each of its three directions into each of the other directions. So each direction gets the same share, one third of the whole, of any quantity that is shared out among the three directions and is unchanged by the symmetries of the carrier. Another book of the collection derives general relativity: *The Gravitational Passage; Deriving General Relativity from the Immanent Metaphysics*. The context instance is the internal space $\mathbb{C}P^2 \times S^2$ of *The Gravitational Passage*, read as a space of possibility, with radii. The context instance has a factor $\mathbb{C}P^2$ that is also the space of the pure states of a quantum system with three alternatives. Chapter 5 uses the symmetry argument again, for the three alternatives. The vacuum relation is the relation that sets the cosmological constant

(the energy density of empty space). On the context instance *The Gravitational Passage* counts the three channels of its vacuum relation, one channel for each of the three alternatives. So the three channels take equal shares.

Chapter 6 sets the Born rule beside the vacuum relation of *The Gravitational Passage*, Chapter 13, and shows the step that joins the Born rule to the vacuum relation: each is an instance of the Incommensuration Theorem. The Born rule is the instance at a single measurement. The vacuum relation is the instance at the scale of the observer's horizon (the observer's particle horizon), which is the boundary of the region from which light has had time to reach the observer.

III. Where Each Result Is Carried

Each result of the collection is set out in full in one book. I say that the book which sets out a result in full "carries" the result. When another book of the collection carries a result, I cite that other book by chapter and section, and I add only what the other book does not carry. One book carries most of the results that this book touches: *The Quantum Bridge; Entanglement, Error Correction, and the Limits of Simulation*. A further book sets out the vacuum relation for general readers: *The View from Inside; On the Cosmological Constant and Why Empty Space Weighs So Little*. Table 0.1 lists the results that the chapters of this book touch, the book that carries each, and what this book adds.

Result	Carried in	What this book adds
The Tsirelson bound on the derived Hilbert space, and the sectors in which it is reached	*The Quantum Passage*, Chapter 8, Section III	The derivation worked step by step (Chapter 1)

Quantum error correction: the conditions of Knill and Laflamme, the quantum Singleton bound, and the distance of a code	*The Quantum Bridge*, Chapter 4, Sections II to V	Nothing
Gravity in the error budget of real hardware	*The Quantum Bridge*, Chapter 5	Nothing
The decoherence of masses, collapse models and the entanglement test	*The Quantum Bridge*, Chapter 6	What the finite algebra contributes to decoherence (Chapter 2)
The Clifford group, stabilizer states, magic states and their distillation	*The Quantum Bridge*, Chapter 4, Section VII	The relation of two measures of magic for one qubit (Chapter 3); the count of non-Clifford gates (Chapter 4)
The reach of simulation and the errors of approximate simulation	*The Quantum Bridge*, Chapters 7 and 8	Nothing
The vacuum relation $\rho_\Lambda = c_1 \, f \, M_{Pl}{}^4/S$ and the count of channels $c_1 = \chi(\mathbb{C}P^2) = 3$	*The Gravitational Passage*, Chapter 13, Sections I and II; *The View from Inside*, Chapters 11 to 13	The equal weight of the three channels, from the symmetry of the three alternatives of the context instance (Chapter 5); the step joining the vacuum relation to the Born rule (Chapter 6)

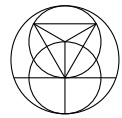
Table 0.1. The results the chapters touch, the book that carries each, and what this book adds.

The count $c_1 = 3$ appears in one relation of the collection's account of physics: the vacuum relation of the observer's particle horizon, which sets the cosmological constant (*The Quantum Bridge*, "What This Book Claims and How It Is Set Out", Section I, and Chapter 1, Section V). In the vacuum relation, c_1 is the count of channels, $c_1 = \chi(\mathbb{C}P^2) = 3$. The symbol

$\chi(\mathbb{C}P^2)$ stands for the Euler characteristic of the complex projective plane $\mathbb{C}P^2$. The Euler characteristic is a topological count of the shape of a space. *The Gravitational Passage* counts the channels on the context instance of the internal space $\mathbb{C}P^2 \times S^2$: the internal space of *The Gravitational Passage*, read as a space of possibility, with radii. The content instance is the space of states of the transcendent carrier. In the content instance the mathematical form of the context instance is read as a space of states with no size (*The Quantum Passage*, Chapter 6, Section V; Chapter 5, Section V of this book). The count c_1 is absent from every error rate, every rate of decoherence and every error of simulation (*The Quantum Bridge*, Chapter 4, Section IV, Chapter 6, Section VI, and Chapter 8, Proposition 8.1). Among the relations of physics, the chapters of this book use c_1 in the vacuum relation and nowhere else.

IV. Summary

I give six pieces of work here that go beyond the derivation of quantum mechanics in *The Quantum Passage*. The first piece is a step-by-step derivation of the Tsirelson bound on the comparison algebra. The second piece is what the finite algebra contributes to decoherence. The third piece is how two measures of magic for one qubit are related. The fourth piece is how many non-Clifford gates a given amount of magic requires. The fifth piece is why the three units of c_1 have equal weight. The sixth piece is the step that joins the Born rule to the vacuum relation. No step of the derivation rests on any of the six pieces. Where *The Quantum Passage*, *The Quantum Bridge* or *The Gravitational Passage* carries a result, this book cites the result in that book and adds only what that book does not carry.



Chapter 1

The Tsirelson Bound from the Comparison Algebra

The Largest CHSH Value, Worked Step by Step on $A_F \otimes A_F$

ABSTRACT

The CHSH inequality is named for Clauser, Horne, Shimony and Holt. The CHSH inequality limits one combination of four correlations between two separated measurements, the CHSH combination. The Tsirelson bound, $2\sqrt{2}$, is the largest value the CHSH combination can take. Cirel'son's theorem states that the CHSH combination never exceeds $2\sqrt{2}$ on any Hilbert space. *The Quantum Passage*, Chapter 8, Section III, states the bound for the Hilbert space derived from the comparison algebra, and says where on $A_F \otimes A_F$ the bound is reached. This chapter works out, step by step, the derivation of the bound on the comparison algebra. Two systems taken together have the composite space $H_F \otimes H_F$, which splits into nine sectors. The CHSH operator maps each sector to itself. So the largest value over all states is the largest over the sectors. An identity of Landau shows that the value in every sector is at most $2\sqrt{2}$: $C^2 = 4I + [A, A'] \otimes [B, B']$. In every sector where either party has the commutative carrier $\mathbb{C}$, the value is at most 2, the classical value. In every sector where both parties have a non-commutative carrier, $\mathbb{H}$ or $M_3(\mathbb{C})$, the value $2\sqrt{2}$ is reached, as the singlet state on $\mathbb{C}^2 \otimes \mathbb{C}^2$, with settings given

> explicitly, shows. A superposition across sectors is a state spread over more than one sector. A superposition across sectors cannot exceed the bound.

I. The Question

The CHSH inequality was published in 1969 by four physicists: John Clauser, Michael Horne, Abner Shimony and Richard Holt. The name CHSH is made of the initials of the four authors. The CHSH inequality limits the correlations that any local hidden-variable theory can produce between two separated measurements. When a theory has two features, we call it a local hidden-variable theory. First, properties carried by each system fix each measurement's result in advance. Second, a choice at one place cannot change a result at the other.

In the setting of the CHSH inequality, each of two parties chooses one of two measurements, and each measurement has the outcomes $+1$ and -1. A correlation is the average, over many runs, of the product of the two parties' outcomes. The four pairs of choices give four correlations, and the CHSH inequality says that one combination S of the four correlations obeys $|S| \leq 2$. For some states and some choices of measurement, quantum mechanics gives values of $|S|$ above 2, but in 1980 the mathematician Boris Cirel'son proved that quantum mechanics never gives $|S|$ a value above $2\sqrt{2} \approx 2.828$. The largest value $2\sqrt{2}$ is named the Tsirelson bound after Cirel'son, whose name is also spelled Tsirelson.

Cirel'son's theorem holds on every Hilbert space. A Hilbert space is the space of states of a quantum system. Each state is a vector of the space, and each measurement acts on the vectors as an operator. In this book I rely on *The Quantum Passage*. In *The Quantum Passage* I derive the Hilbert space, the observables on the Hilbert space and the tensor product of Hilbert spaces from the comparison algebra (Chapters 5, 6 and 8 of *The Quantum Passage*). So the Tsirelson bound follows from the derivation in *The Quantum Passage*. One question about the comparison algebra re-

mains: where on $A_F \otimes A_F$ the bound is reached. $A_F \otimes A_F$ is the algebra of two systems taken together. Here I give the answer, with every step of the derivation written out.

The answer to where the bound is reached also answers a second question. The second question is why the largest value is $2\sqrt{2}$ and not 4. In 1994 the physicists Sandu Popescu and Daniel Rohrlich showed that 4 is the largest value that the impossibility of signaling alone allows. Signaling would mean that one party sends a message to the other party by the choice of a measurement. The bound $2\sqrt{2}$ holds on the derived Hilbert space, so the quantum theory derived from comparison predicts no value above $2\sqrt{2}$.

II. The Algebra and Its Sectors

Chapter 5 of *The Quantum Passage* derives the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from the three axioms of *The Quantum Passage*. In the formula for A_F, $\mathbb{C}$ is the algebra of complex numbers, $\mathbb{H}$ is the algebra of quaternions, and $M_3(\mathbb{C})$ is the algebra of 3×3 complex matrices. The sign $\oplus$ marks a direct sum: an element of A_F is a triple, one element from each of the three algebras, and the three parts act separately and never mix.

The derivation of A_F is a chain with no continuous parameter. A continuous parameter is a number free to take any value in a range. The chain begins with the algebraic encoding of comparison and the finite dimension of the internal space, and continues through semisimplicity, the Artin-Wedderburn decomposition, the Three-Carrier Theorem and the labeling of the carriers. The chain ends with re-projection, the step that fixes the rank of $M_3(\mathbb{C})$ at three.

I call each of the three algebras a carrier. The data below for the three carriers are from Chapter 6, Section III, of *The Quantum Passage*. A carrier's module is the space of vectors on which the carrier acts. The ray space of a module is the space of its states. In the ray space, two vectors that differ

only by a complex factor count as one state. The unitary group of a carrier is the group of its elements that keep the length of every vector unchanged.

The omniscient carrier is $\mathbb{C}$, the complex numbers, whose module is $\mathbb{C}$ itself, with one complex dimension. The unitary group of $\mathbb{C}$ is $U(1)$, the complex numbers of size one.

The immanent carrier is the quaternion algebra $\mathbb{H}$. The module of $\mathbb{H}$ is $\mathbb{C}^2$, the qubit: the vector space of a quantum system with two levels. The ray space of $\mathbb{C}^2$ is the Bloch sphere S^2, and the unit quaternions form the group $SU(2)$. When complex multiples are allowed, the quaternions generate $M_2(\mathbb{C})$, the algebra of 2×2 complex matrices.

The transcendent carrier is $M_3(\mathbb{C})$, the algebra of 3×3 complex matrices. The module of $M_3(\mathbb{C})$ is $\mathbb{C}^3$, the space of the three colors, which are the three charges of the strong force. The ray space of $\mathbb{C}^3$ is the complex projective plane $\mathbb{C}P^2$. The unitary group of $M_3(\mathbb{C})$ is $U(3)$. Unimodularity reduces $U(3)$ to $SU(3)$, the 3×3 unitary matrices of determinant one.

The finite Hilbert space H_F is the direct sum of the three modules $\mathbb{C}$, $\mathbb{C}^2$ and $\mathbb{C}^3$.

$$H_F = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 = \mathbb{C}^6 \qquad (1.1)$$

An observable is the operator for a quantity that a measurement reads. The observables of a measurement on H_F are the self-adjoint elements of the complex algebra that A_F generates on H_F, which is $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$. A self-adjoint element is an element that equals its own conjugate transpose, so a self-adjoint element has real eigenvalues, which are the possible readings of the measurement. $\mathbb{C}$ and $M_3(\mathbb{C})$ are already complex algebras, so the complex algebra that each generates is the algebra itself. For $\mathbb{H}$ the complex algebra also contains the Pauli matrices, which are the imaginary quaternions multiplied by i. An observable on H_F is therefore a triple (z, q, M), in which z is a real number, q is a self-adjoint 2×2 com-

plex matrix, and M is a self-adjoint 3 × 3 complex matrix. The observable (z, q, M) acts on H_F as the direct sum z ⊕ q ⊕ M: as the number z on $\mathbb{C}$, as the matrix q on $\mathbb{C}^2$, and as the matrix M on $\mathbb{C}^3$.

III. The Composite and Its Nine Sectors

When each of two separated systems is described by A_F, the tensor product $A_F \otimes A_F$ describes the two systems together. The tensor product $A_F \otimes A_F$ acts on $H_F \otimes H_F$. The tensor product is the way quantum mechanics joins two systems into one, so the dimension of the joint space is the product of the dimensions of the spaces of the two systems. A C*-algebra is an algebra of operators with an adjoint and a norm, of the kind that quantum observables form. The tensor product of two C*-algebras is again a C*-algebra. The GNS construction is named for Gelfand, Naimark and Segal. For a C*-algebra and a state of the algebra, the GNS construction produces a Hilbert space on which the algebra acts. On $A_F \otimes A_F$, the GNS construction produces $H_F \otimes H_F$, the tensor product of the Hilbert spaces of the two systems (*The Quantum Passage*, Chapter 6, Section VI, Postulate 4). The tensor product distributes over direct sums, the way multiplication distributes over addition. So $H_F \otimes H_F$ splits into the nine pieces of (1.2).

$$H_F \otimes H_F = \oplus_{i,j} (\mathbb{C}^i \otimes \mathbb{C}^j), \quad i, j \in \{1, 2, 3\} \qquad (1.2)$$

The space $H_F \otimes H_F$ has dimension 36, and the nine pieces of (1.2) are called its sectors. Carrier 1 is $\mathbb{C}$, carrier 2 is $\mathbb{H}$, and carrier 3 is $M_3(\mathbb{C})$. Carrier i has the module $\mathbb{C}^i$, because the modules of $\mathbb{C}$, $\mathbb{H}$ and $M_3(\mathbb{C})$ are $\mathbb{C}$, $\mathbb{C}^2$ and $\mathbb{C}^3$. The sector (i, j) is the piece $\mathbb{C}^i \otimes \mathbb{C}^j$, the part of the joint space in which the first system lies in the module of carrier i and the second system lies in the module of carrier j. The sector (i, j) has dimension ij: the sector (1, 1) has dimension 1; the sectors (1, 2) and (2, 1) have dimension

2; the sectors (1, 3) and (3, 1) have dimension 3; the sector (2, 2) has dimension 4; the sectors (2, 3) and (3, 2) have dimension 6; and the sector (3, 3) has dimension 9.

An observable $A = (z, q, M)$ acts on H_F as $z \oplus q \oplus M$, one component on each module, so the product $A \otimes B$ acts on the sector (i, j) as the i-th component of A tensored with the j-th component of B. Hence $A \otimes B$ maps each sector to itself: written as a matrix, $A \otimes B$ has no block joining two different sectors. A sum of products of the form $A \otimes B$ also maps each sector to itself, and the CHSH operator of Section IV is a sum of products of that form.

IV. The CHSH Operator

Two parties measure two separated systems, one system each, and physicists call the two parties Alice and Bob. Alice chooses between the observables A and A′, and Bob chooses between the observables B and B′. Each of the four observables has the eigenvalues $+1$ and -1 only. So each of the four observables squares to the identity: $A^2 = I$, sector by sector. Section II writes each observable as a triple (z, q, M). For an observable whose only eigenvalues are $+1$ and -1, the triple has three properties. First, $z = \pm 1$. Second, q is $\pm I$, or $q = \hat{u} \cdot \sigma$ for a unit vector $\hat{u}$ in $\mathbb{R}^3$, with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices. Third, every eigenvalue of M is $+1$ or -1. The observable $\hat{u} \cdot \sigma$ measures spin along the direction $\hat{u}$.

For a state $|\psi\rangle$ of $H_F \otimes H_F$, write $E(a, b)$ for the correlation when Alice measures A and Bob measures B. Write $E(a, b')$, $E(a', b)$ and $E(a', b')$ in the same way for the other three pairs of choices. Each correlation is an expectation value, the average reading over many runs: $E(a, b) = \langle\psi|\ A \otimes B\ |\psi\rangle$. The CHSH combination S is given by equation (1.3).

$$S = E(a, b) - E(a, b') + E(a', b) + E(a', b') \qquad (1.3)$$

S is the expectation value of one operator. The operator is C, the CHSH operator of equation (1.4).

$$C = A \otimes B - A \otimes B' + A' \otimes B + A' \otimes B' = A \otimes (B - B') + A' \otimes (B + B') \quad (1.4)$$

V. The Sectors Separate

The CHSH operator C is a sum of products of the form $A \otimes B$, so by Section III, C maps each sector to itself. Write C_{ij} for the restriction of C to the sector (i, j): C acting on that sector alone. Any state can be split into pieces, one in each sector. The split is $|\psi\rangle = \Sigma_{ij}\, c_{ij}\, |\psi_{ij}\rangle$, with each $|\psi_{ij}\rangle$ a unit vector of its sector. For a state split in that way, the expectation value of C is given by equation (1.5).

$$\langle\psi|\, C\, |\psi\rangle = \Sigma_{ij}\, |c_{ij}|^2\, \langle\psi_{ij}|\, C_{ij}\, |\psi_{ij}\rangle \quad (1.5)$$

In $\langle\psi|\, C\, |\psi\rangle$, every term that pairs two different sectors is zero, because C has no block between sectors, so equation (1.5) holds. The weights $|c_{ij}|^2$ sum to one, so the expectation value of C is a convex combination of the expectation values in the sectors. A convex combination is a weighted average whose weights are zero or more and sum to one. For a self-adjoint operator such as C_{ij}, the operator norm $\|C_{ij}\|$ is the largest size of its expectation value over the unit vectors of the sector. The largest |S| over all states is therefore the largest operator norm $\|C_{ij}\|$ over the sectors, and |S| equals that largest value for a state with all its weight in a best sector. So the search for the largest |S| comes down to the norm of the CHSH operator in each sector separately.

VI. The Upper Bound

The upper bound follows from an identity, an equation that holds in every sector and uses only the facts $A^2 = A'^2 = I$ and $B^2 = B'^2 = I$. Square C in its second form in (1.4) to get equation (1.6).

$$C^2 = A^2 \otimes (B - B')^2 + A'^2 \otimes (B + B')^2 + AA' \otimes (B - B')(B + B') + A'A \otimes (B + B')(B - B') \quad (1.6)$$

Bob's two observables satisfy $B^2 = B'^2 = I$, so $(B - B')^2 = 2I - \{B, B'\}$ and $(B + B')^2 = 2I + \{B, B'\}$. The bracket $\{B, B'\} = BB' + B'B$ is called the anticommutator of B and B′. The first two terms of (1.6) therefore sum to $4(I \otimes I)$, because $A^2 = A'^2 = I$ as well. Also $(B - B')(B + B') = [B, B']$ and $(B + B')(B - B') = -[B, B']$, where $[B, B'] = BB' - B'B$ is the commutator of B and B′. The commutator of B and B′ is zero exactly when B and B′ commute ($BB' = B'B$). The last two terms of (1.6) therefore sum to $(AA' - A'A) \otimes [B, B']$. Adding the two sums gives equation (1.7).

$$C^2 = 4(I \otimes I) + [A, A'] \otimes [B, B'] \qquad (1.7)$$

Lawrence Landau, a mathematical physicist, published the identity (1.7) in 1987. Landau used the identity to obtain Cirel'son's bound. *The Quantum Passage*, Chapter 8, Section III, writes the CHSH operator with B′ replaced by −B′, as $C = A(B + B') + A'(B - B')$, and in that form the identity reads $C^2 = 4 - [A, A'][B, B']$. The bound is the same in both forms, because −B′ is also an observable whose only eigenvalues are +1 and −1.

For self-adjoint operators X and Y of norm at most one, inequality (1.8) holds.

$$\|[X, Y]\| = \|XY - YX\| \leq \|XY\| + \|YX\| \leq 2\,\|X\|\,\|Y\| \leq 2 \qquad (1.8)$$

The first step of (1.8) is the triangle inequality, which every norm obeys, and the second step is the rule that the norm of a product is at most the product of the norms. Each of A, A′, B and B′ has norm one, because its eigenvalues are +1 and −1, so $\|[A, A']\| \leq 2$ and $\|[B, B']\| \leq 2$. By (1.7), the norm of C^2 then obeys inequality (1.9).

$$\|C^2\| \leq 4 + \|[A, A']\|\,\|[B, B']\| \leq 8 \qquad (1.9)$$

The operator C is self-adjoint. So $\|C\| = \|C^2\|^{1/2} \leq \sqrt{8} = 2\sqrt{2}$. Every state therefore gives $|\langle\psi|\, C\, |\psi\rangle| \leq 2\sqrt{2}$. The inequality $|S| \leq 2\sqrt{2}$, for every state, is the Tsirelson bound.

The bound in every sector and on every space. The computation of the bound used two facts only: that each observable squares to the identity, and that the operator norm is submultiplicative, which means that the norm of a product is at most the product of the norms. Both facts hold in every dimension, so the bound $2\sqrt{2}$ holds in every sector of the composite: in the sectors $\mathbb{C} \otimes \mathbb{C}$, $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^3 \otimes \mathbb{C}^3$ and in every cross sector alike. The bound also holds on the full Hilbert space of the spectral triple (*The Quantum Passage*, Chapter 6, Section V), just as the bound holds on H_F. The spectral triple joins an algebra, its Hilbert space and a Dirac operator into one structure.

VII. The Bound Is Reached

One example is enough to show that the bound is reached: one state and one choice of A, A′, B and B′ with $|S| = 2\sqrt{2}$. Take the sector $(2, 2)$: the module $\mathbb{C}^2 \otimes \mathbb{C}^2$ of $\mathbb{H} \otimes \mathbb{H}$, which is a pair of qubits. In the sector $(2, 2)$, each party has two kinds of observables with eigenvalues ± 1: the observables $\pm I$, and the observables $\hat{u} \cdot \sigma$ for unit vectors $\hat{u}$. Choose the measurement directions of (1.10), in the plane of $\hat{x}$ and $\hat{z}$, the unit vectors along the x axis and the z axis.

$$\text{Alice: } \hat{a} = \hat{z}, \quad \hat{a}' = \hat{x}; \quad \text{Bob: } \hat{b} = (\hat{z} + \hat{x})/\sqrt{2}, \quad \hat{b}' = (\hat{x} - \hat{z})/\sqrt{2} \tag{1.10}$$

With the directions of (1.10), $A = \sigma_3$, $A' = \sigma_1$, $B = (\sigma_3 + \sigma_1)/\sqrt{2}$ and $B' = (\sigma_1 - \sigma_3)/\sqrt{2}$. Take the singlet state of equation (1.11).

$$|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \tag{1.11}$$

In the singlet state, $E(a, b) = \langle\psi| (\hat{a} \cdot \sigma) \otimes (\hat{b} \cdot \sigma) |\psi\rangle = -\hat{a} \cdot \hat{b}$. The correlation $-\hat{a} \cdot \hat{b}$ is minus the cosine of the angle between Alice's direction and Bob's direction. When Alice and Bob measure along the same direction, the correlation is -1, so in the singlet state readings along a common direction are always opposite. For the directions of (1.10), the four correlations of (1.3) take the values in (1.12).

$$E(a, b) = -1/\sqrt{2}, \quad E(a, b') = +1/\sqrt{2}, \quad E(a', b) = -1/\sqrt{2}, \quad E(a', b') = -1/\sqrt{2} \qquad (1.12)$$

For example $\hat{a} \cdot \hat{b} = 1/\sqrt{2}$, and so $E(a, b) = -1/\sqrt{2}$. Putting the four correlations into (1.3) gives equation (1.13).

$$S = -1/\sqrt{2} - 1/\sqrt{2} - 1/\sqrt{2} - 1/\sqrt{2} = -4/\sqrt{2} = -2\sqrt{2} \qquad (1.13)$$

So $|S| = 2\sqrt{2}$, and the bound is reached.

> **Why these settings.** On $\mathbb{C}^2$ the commutator of two spin observables is $[\hat{a} \cdot \sigma, \hat{a}' \cdot \sigma] = 2i(\hat{a} \times \hat{a}') \cdot \sigma$, of norm $2|\hat{a} \times \hat{a}'|$. The norm $2|\hat{a} \times \hat{a}'|$ equals 2 exactly when the two axes are perpendicular. Alice's axes $\hat{z}$ and $\hat{x}$ are perpendicular, and so are Bob's axes, since $(\hat{z} + \hat{x}) \cdot (\hat{x} - \hat{z}) = 0$. For the settings of (1.10), $\hat{z} \times \hat{x} = \hat{y}$ and $\hat{b} \times \hat{b}' = \hat{y}$, so $[A, A'] = [B, B'] = 2i\sigma_2$, and $[A, A'] \otimes [B, B'] = -4\, \sigma_2 \otimes \sigma_2$. The singlet satisfies $(\sigma_2 \otimes \sigma_2)|\psi\rangle = -|\psi\rangle$, so by (1.7) $C^2|\psi\rangle = (4 + 4)|\psi\rangle = 8|\psi\rangle$. The singlet is therefore a state on which C^2 takes its largest value, 8. On the singlet, the expectation of C has magnitude $\sqrt{8} = 2\sqrt{2}$.

VIII. Sector by Sector

Five sectors have $\mathbb{C}$ on at least one side: (1, 1), (1, 2), (2, 1), (1, 3) and (3, 1). In each of the five sectors, at least one party measures on the one-dimensional module of $\mathbb{C}$. On the module of $\mathbb{C}$, every observable with eigenvalues ± 1 is the number $+1$ or -1, so the two observables of the party on $\mathbb{C}$ commute, and their commutator is zero. By (1.7), $C^2 = 4I$ in each of the five sectors, so $\|C\| = 2$. The value 2 is the classical value: the bound $|S| \leq 2$ that every local hidden-variable theory obeys.

In the sector (2, 2), Section VII shows a state and settings with $|S| = 2\sqrt{2}$.

By Section VI, the value in no sector exceeds $2\sqrt{2}$. In the sectors (2, 3), (3, 2) and (3, 3), the value $2\sqrt{2}$ is also reached. Restrict $M_3(\mathbb{C})$ to the span of the first two basis vectors of $\mathbb{C}^3$ to reach $2\sqrt{2}$ in those three sectors. Take

an observable of the form $q \oplus (\pm 1)$, in which q is a 2×2 observable with eigenvalues ± 1 acting on the span of the first two basis vectors, and the number $+1$ or -1 acts on the third basis vector. The observable $q \oplus (\pm 1)$ then has eigenvalues ± 1. Take a state that lies in the two-dimensional span on each side where $M_3(\mathbb{C})$ acts, and use the settings of (1.10) for q. For that state and the settings of (1.10), the steps of Section VII apply unchanged, and $|S| = 2\sqrt{2}$ again. So the value $2\sqrt{2}$ is reached in each of the four sectors $(2, 2)$, $(2, 3)$, $(3, 2)$ and $(3, 3)$. The four sectors are the sectors where both parties have a non-commutative carrier. No state in the four sectors gives a value above $2\sqrt{2}$.

A superposition across sectors is a state spread over several sectors. By (1.5), the value of a superposition across sectors is a convex combination of the values in the sectors over which the superposition is spread. A convex combination is never larger than the largest of its terms. Every term is at most $2\sqrt{2}$, so no superposition across sectors exceeds the bound.

The results of this section agree with the statement of *The Quantum Passage*, Chapter 8, Section III: "On $A_F \otimes A_F$ the classical bound 2 holds in every sector in which $\mathbb{C}$ appears on either side, and $2\sqrt{2}$ is reached in every sector in which both sides carry a non-commutative carrier."

IX. Why Non-Commutativity Carries the Bound

The identity (1.7) shows that the commutators of each party's two observables set how far the CHSH value can rise above 2. Where the commutator of a party's two observables is zero, as on the commutative carrier $\mathbb{C}$, the omniscient carrier, the largest value stays at 2. Where the commutator is not zero, as on the non-commutative carriers $\mathbb{H}$, the immanent carrier, and $M_3(\mathbb{C})$, the transcendent carrier, the value can exceed 2, but by (1.8) the value is at most $2\sqrt{2}$.

On both non-commutative carriers the largest value is the same, $2\sqrt{2}$, because the CHSH combination uses one pair of two-outcome observables for each party. In every dimension, the commutator of two two-outcome

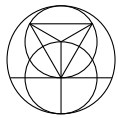
observables has a norm of at most 2. The norm 2 is already reached on a two-dimensional subspace. $M_3(\mathbb{C})$ is larger than $\mathbb{H}$: $M_3(\mathbb{C})$ has three colors and eight generators. But the larger size of $M_3(\mathbb{C})$ produces no stronger correlation. The qubit is the module of $\mathbb{H}$, and the qubit is the smallest space on which each party can reach the bound.

The derivation in *The Quantum Passage* shows why the comparison algebra has non-commutative carriers. *The Quantum Passage*, Chapter 11, Section VIII, states the reason: the algebra has two non-commutative carriers, "$M_3(\mathbb{C})$, which the continuously asymmetric mode requires, since an observable that commutes with the Dirac operator without being central exists only in a non-commutative algebra, and $\mathbb{H}$, which the carrier condition requires; and its one commutative carrier gives back the classical bound 2". An element of an algebra is central when the element and every element of the algebra commute. The algebraic argument is in *Non-Commutative Geometry and the Immanent Metaphysics*, in the section titled "Continuity Requires Non-Commutativity" (Chapter 5, Section II). The Incommensuration Theorem requires the non-commutative carriers. The non-commutative carriers allow values above 2, and the norm inequality (1.8) limits the values to at most $2\sqrt{2}$. So the size of the quantum correlations follows from the structure of comparison, and so does their existence.

X. Summary

The Tsirelson bound $2\sqrt{2}$ follows from the comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ in four steps. First, the GNS construction on the algebra produces the six-dimensional space H_F. Second, two systems together have the algebra $A_F \otimes A_F$, acting on $H_F \otimes H_F$. Third, the CHSH operator maps each of the nine sectors of $H_F \otimes H_F$ to itself, so the largest value over all states is the largest value over the sectors. Fourth, Landau's identity applied in each sector shows that the value in every sector is at most $2\sqrt{2}$. In every sector where either party has the commutative carrier $\mathbb{C}$, the

classical value 2 holds. In every sector where both parties have a non-commutative carrier, $\mathbb{H}$ or $M_3(\mathbb{C})$, the value $2\sqrt{2}$ is reached. On the singlet on $\mathbb{C}^2 \otimes \mathbb{C}^2$, with the settings (1.10), $|S|$ equals $2\sqrt{2}$. C^2 has the eigenvalue 8 on the singlet, which shows that $|S|$ equals $2\sqrt{2}$. No superposition across sectors exceeds $2\sqrt{2}$. The bound follows from the algebra that the three axioms of *The Quantum Passage* determine. No step of the derivation postulates the bound or imports the bound from outside the derivation.



Chapter 2

Decoherence and the Comparison Algebra

What the Finite Algebra Supplies, What the Continuum Supplies, and Where the Time Scale Comes From

ABSTRACT

The comparison algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ supplies a qubit and a finite Dirac operator. The qubit is $\mathbb{C}^2$, the module of the quaternions $\mathbb{H}$: the space on which the quaternions act. The finite Dirac operator couples the $\mathbb{H}$ carrier to the $\mathbb{C}$ carrier through the Yukawa matrices, and it never touches color, the charge on which the carrier $M_3(\mathbb{C})$ acts. A coupling between two finite sectors is coherent. The coupling mixes the two sectors reversibly, at a frequency set by the coupling, and destroys no coherence. Decoherence needs an environment with many degrees of freedom and a continuous spectrum, an environment which in the product geometry M $\times$ F is the continuum of the spacetime factor. When a qubit is coupled to the continuum by $H_{int} = \sigma_z \otimes B$, the coherence of the qubit decays at the rate $\Gamma = (4/\hbar^2) \int_0^\infty \mathrm{Re}\ \langle B(\tau)B(0)\rangle\, d\tau$. For a correlation function of strength g^2 and correlation time τ_c, the rate is $\Gamma = 4g^2\tau_c/\hbar^2$. The time scale τ_c belongs to the environment, not the finite algebra. The infidelity of the resulting channel is an output of the dynamics. The formula for Γ is an instance, worked from the algebra, of the statement of *The Quantum Bridge* that the rate of decoherence is set by the coupling of a system to what registers its alternatives.

I. What the Finite Algebra Supplies

The finite algebra, the comparison algebra A_F, has three carriers. The carrier $\mathbb{C}$ is the omniscient carrier. The carrier $\mathbb{H}$ is the immanent carrier, and its module $\mathbb{C}^2$ is a qubit. The carrier $M_3(\mathbb{C})$ is the transcendent carrier, and it acts on color. Color is the charge that quarks carry for the strong force. The Standard Model is the established theory of the known particles and of every force except gravitation. The finite geometry of the Standard Model is the part of the Standard Model that the finite algebra and a finite Dirac operator describe. In the finite geometry, the Dirac operator D_F is made of the Yukawa matrices and of a Majorana entry. The Yukawa matrices couple left-handed doublets, on which the carrier $\mathbb{H}$ acts, to right-handed singlets, on which the carrier $\mathbb{C}$ acts. A doublet is a pair of states that $\mathbb{H}$ mixes with each other, and a singlet is a single state. "Left-handed" and "right-handed" name the two kinds of handedness that a particle of matter can have. The Majorana entry would join a right-handed neutrino to its own antiparticle. Under the two-terms postulate of *Non-Commutative Geometry and the Immanent Metaphysics*, the Majorana entry is zero (its Proposition 8.4(i) and Chapter 19, Section X). The operator D_F commutes with the color carrier, which means that applying D_F and then an element of the color carrier gives the same result as applying the element first and D_F second. In symbols,

$$[D_F, (0, 0, m)] = 0 \qquad \text{for every } m \text{ in } M_3(\mathbb{C}) \qquad (2.1)$$

Here $(0, 0, m)$ is the element of the comparison algebra with m in the color carrier and zero in the other two carriers. Equation (2.1) is Proposition 8.4(ii) of *Non-Commutative Geometry and the Immanent Metaphysics*. Picture the three carriers of the finite algebra as the corners of a triangle, the triangle of carriers, with an edge between each pair. D_F acts along one edge of the triangle of carriers, the edge between the immanent and omniscient carriers. On that edge D_F generates the masses of the fermions, as Section

II shows. Proposition 8.4(iii) of *Non-Commutative Geometry and the Immanent Metaphysics* states: "The transcendent carrier is coupled to no other carrier".

When D_F is written as a matrix, a block is a group of entries that takes the states of one carrier to the states of another. D_F has no block joining $\mathbb{H}$ to $M_3(\mathbb{C})$, because the transcendent carrier is coupled to no other carrier. Quarks carry both weak isospin, the charge of the weak force on which $\mathbb{H}$ acts, and color. Quarks carry both charges because quarks occupy a sector (a part of the space of particle states) on which $\mathbb{H}$ acts from the left and $M_3(\mathbb{C})$ acts from the right, and not because D_F joins $\mathbb{H}$ to $M_3(\mathbb{C})$.

II. Coupling Between Finite Sectors Is Coherent

The coupling that D_F provides joins a doublet state to a singlet state with a strength y. The doublet states and the singlet states lie in two finite sectors, two parts of the finite space of states. The strength y is a Yukawa coupling times the scale of the Higgs field. A Yukawa coupling is an entry of a Yukawa matrix. The scale of the Higgs field is the size of the constant value that the Higgs field takes throughout space. When the coupling of D_F joins two finite sectors, the two sectors form a closed system of finite dimension, and the evolution of a closed system of finite dimension is unitary, which means reversible (*The Quantum Passage*, Chapter 6, Section VI). Take two states of equal energy, a first state and a second state, joined with strength y, and let the system start in the first state. An amplitude is the number whose squared size gives the probability of finding the system in a given state. The symbol $\hbar$ denotes the reduced Planck constant. At time t the amplitude of the first state is $\cos(yt/\hbar)$ and the amplitude of the second state is $-i\sin(yt/\hbar)$, so the system passes from the first state to the second and back. The system returns to the first state, up to a sign, each time a further interval $\pi\hbar/y$ has passed. Nothing is lost: the qubit's coherence passes into the joint state of the doublet and singlet sectors and

comes back. In the Standard Model, the mass of a fermion is the coherent join of a doublet state to a singlet state. A fermion is a particle of matter, such as a quark. The mass eigenstate is the state with one definite mass, and it combines the doublet component and the singlet component. A finite coupling decoheres nothing, because a finite system has no environment, no surroundings into which its coherence could escape.

III. Where the Environment Comes From

The environment of a system is its surroundings. The degrees of freedom of the environment are the independent ways in which the environment can change. A continuous spectrum is an unbroken range of energies. An irreversible loss of coherence needs an environment with many degrees of freedom and a continuous spectrum, so that correlations between the system and the environment spread into the environment and do not return. The product geometry $M \times F$ joins two factors: the manifold M, which is spacetime, and the finite geometry F. The finite algebra and D_F describe the finite geometry F. In the product geometry $M \times F$, the finite factor carries the internal degrees of freedom, such as weak isospin and color (*Non-Commutative Geometry and the Immanent Metaphysics*, Chapter 13, Section I). The manifold factor carries the continuum. The continuum consists of the spinor fields of $L^2(M, S)$, which are the fields of the particles of matter, and the gauge fields on spacetime, which are the fields of the forces, with their continuous spectra. The continuum of the manifold factor is the continuously asymmetric mode of *The Quantum Passage* (its Chapter 4, Section III), a mode that is continuous and not symmetric, and the continuum is where an environment comes from. Decoherence of the qubit is the passing of its coherence into the continuum.

IV. The Rate and Its Time Scale

Let the qubit couple to its environment by $H_{int} = \sigma_z \otimes B$. Here H_{int} is the energy of the qubit's interaction with the environment. The Pauli matrix σ_z has the value $+1$ on one basis state of the qubit and -1 on the other. B is an operator of the environment, of the dimension of an energy, whose correlation function is $C(\tau) = \langle B(\tau)B(0)\rangle$. The correlation function measures how far the value of B at one moment is still related to its value a time τ later, and the angle brackets denote an average over the states of the environment. An environment's correlation time is the time over which $C(\tau)$ dies away. The state of the qubit is a matrix ρ. The diagonal of ρ holds the probabilities of the qubit's two basis states, and the off-diagonal elements carry the coherence between them. The two basis states of the qubit acquire opposite phases from B, so the off-diagonal element of the qubit's state is multiplied by the average of $\exp(-i\varphi)$, with φ the difference between the two phases. At time t the phase difference is $\varphi(t) = (2/\hbar)\int_0^t B(s)\, ds$. A Gaussian environment is an environment whose fluctuations follow the bell-shaped Gaussian distribution. For a Gaussian environment whose correlation time is short compared with t, the average of $\exp(-i\varphi)$ is $\exp(-\langle\varphi^2\rangle/2)$. The mean square $\langle\varphi^2\rangle$ grows linearly in t. So the coherence decays as $e^{-\Gamma t}$, at the rate

$$\Gamma = (4/\hbar^2) \int_0^\infty \mathrm{Re}\, C(\tau)\, d\tau \qquad (2.2)$$

Take a correlation function $C(\tau) = g^2\, e^{-|\tau|/\tau_c}$, with g an energy and τ_c a time, so that $C(\tau)$ dies away over the correlation time τ_c. For this correlation function the rate is

$$\Gamma = 4g^2\tau_c/\hbar^2 \qquad (2.3)$$

The rate has the dimension of an inverse time. The reason is that $\hbar$ is an energy times a time. So $g^2\tau_c/\hbar^2$ is an energy squared times a time, divided by an energy squared times a time squared. The correlation time τ_c is a property of the environment. No quantity of the finite algebra supplies the correlation time.

The Lindblad equation is the general equation for a quantum system that loses coherence to its surroundings, and the physicists Lindblad, and Gorini, Kossakowski and Sudarshan, found its general form in 1976. Breuer and Petruccione set the equation out in their book *The Theory of Open Quantum Systems*. Pure dephasing is a loss of coherence that leaves the probabilities of the basis states as they are. On times long compared with τ_c, the dynamics of the qubit coupled to B is the Lindblad equation of pure dephasing (Lindblad 1976; Gorini, Kossakowski and Sudarshan 1976; Breuer and Petruccione 2002). For pure dephasing the Lindblad equation is

$$d\rho/dt = -(i/\hbar)[H, \rho] + (\Gamma/2)(\sigma_z \rho \sigma_z - \rho) \qquad (2.4)$$

Here H is the Hamiltonian of the qubit, the operator of the qubit's own energy. The populations are the probabilities of the basis states that σ_z defines. The populations are unchanged, since $\sigma_z\rho\sigma_z$ leaves the diagonal of ρ as it is. The coherence decays at the rate Γ, since $\sigma_z\rho\sigma_z$ reverses the sign of the off-diagonal elements, which carry the coherence. The purity of a state is 1 for a pure state and smaller for a mixed state. A balanced state of the qubit has equal weight on both basis states. The purity of an initially balanced state falls from 1 toward ½ as $\frac{1}{2}(1 + e^{-2\Gamma t})$.

V. The Infidelity of the Channel Is an Output

A channel is the map that takes the state of the qubit at the start to its state at a later time. After a time t the qubit has passed through a dephasing channel that multiplies its coherence by $\lambda = e^{-\Gamma t}$. For a pure input, the fidelity of a channel measures how close the output state is to the input

state. The fidelity is 1 when the output is the input. Over pure inputs, the least fidelity of the dephasing channel is $F_{min} = (1 + \lambda)/2$, reached on the balanced states. The infidelity of the channel is $\frac{1}{2}(1 - F_{min})$, which at time t is

$$\varepsilon(t) = (1 - e^{-\Gamma t})/4 \qquad (2.5)$$

The infidelity rises from 0 to ¼. The value ¼ is the infidelity of the completely depolarizing channel, which replaces every input with the fully mixed state. The infidelity is fixed by the dynamics, through Γ, and is not a further parameter of the rate.

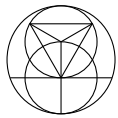
VI. The Agreement with The Quantum Bridge

Another book of this collection is *The Quantum Bridge; Entanglement, Error Correction, and the Limits of Simulation*. *The Quantum Bridge* states the general result in its Chapter 6, Proposition 6.1: "A superposition ends only through a finding, and the rate at which findings are made is set by the coupling of the system to whatever registers its alternatives." A superposition is a quantum state holding more than one alternative at once. In *The Quantum Bridge*, a finding is the registering of one alternative. *The Quantum Bridge* adds, in Section VI of its Chapter 6: "The account has no decoherence rate of its own, no added channel, and no constant that multiplies a rate of the ordinary kind." In the quoted sentence, "the account" is the account of physics set out in this collection of books. I have worked out an instance of the result of Proposition 6.1 from the comparison algebra. The finite algebra adds no decoherence of its own, because its one coupling, the coupling of D_F, is coherent, as Section II shows. The rate (2.2) is set by the coupling of the qubit to the continuum and by the correlation time of the continuum. The color carrier contributes no factor to the rate, because D_F, which commutes with the color carrier, never couples it to another carrier.

Chapter 6 of *The Quantum Bridge* treats the decoherence of masses, the time-dilation decoherence of composite bodies, the lifetimes of the collapse model of the physicists Diósi and Penrose, and the entanglement test. Time-dilation decoherence is the loss of coherence that a composite body suffers because time runs at different rates at different heights. In the collapse model of Diósi and Penrose, a superposition of a mass in two places ends of its own accord after a lifetime. Chapter 5 of *The Quantum Bridge* treats the gravitational entries of the error budget of real hardware, which is the list of the sources of error in a working quantum computer, with the size of each. For a given qubit, the strength g and the correlation time τ_c of (2.3) are properties of the qubit's surroundings, computed from the hardware as Chapter 5 of *The Quantum Bridge* computes the entries of its error budget.

VII. Summary

The finite algebra supplies the qubit, as the module of the quaternions. The finite algebra also supplies a Dirac operator that joins the $\mathbb{H}$ and $\mathbb{C}$ carriers coherently and never touches color. A finite coupling decoheres nothing. The environment that makes decoherence irreversible is the continuum of the spacetime factor. The rate of decoherence is $\Gamma = (4/\hbar^2)\int \mathrm{Re}\, C(\tau)\,d\tau$, and it carries a time scale that belongs to the continuum: the correlation time τ_c. The resulting channel has the infidelity $(1 - e^{-\Gamma t})/4$, and that infidelity is an output of the dynamics. The rate is set by the coupling of the qubit to the continuum, as *The Quantum Bridge*, Proposition 6.1, states. No constant of the comparison algebra multiplies the rate.



Chapter 3

Magic and Stabilizer Infidelity of One Qubit

Two Measures and the Geometry of the Stabilizer Octahedron

ABSTRACT

This chapter compares two measures of how far the state of one qubit lies from the states that Clifford circuits can prepare. A qubit is a quantum system with two alternatives. A Clifford circuit is a circuit built from three basic gates, called the Clifford gates. The states that Clifford circuits prepare are called stabilizer states. The first measure is the log-robustness of magic, $M = \log_2 R$, the logarithm of the robustness R. The robustness R is the least total weight of any way of writing the state as a combination of stabilizer states. The second measure is the stabilizer infidelity, $\delta = \frac{1}{2}(1 - F_S)$, where the stabilizer fidelity F_S is the largest overlap of the state with a stabilizer state. Every state of one qubit has a Bloch vector r: an arrow that places the state in a ball of radius one, with length one for a pure state, which is a state described as completely as quantum mechanics allows. One qubit has six pure stabilizer states, which sit at the corners of an octahedron inside the ball, called the stabilizer octahedron. For a pure state with Bloch vector r outside the stabilizer octahedron, the robustness is the sum of the absolute values of the components of r, while the stabilizer fidelity is $(1 + m)/2$, where m is the largest absolute component of r. So each of the two measures is an exact function of where the Bloch vector lies relative to the stabilizer octahedron. For pure states whose Bloch vectors lie directly above an edge of the octahedron, each measure de-

termines the other. The relation along an edge is $M = \log_2(1 - 4\delta + \sqrt{(8\delta(1 - 2\delta))})$. The T-state is the state that the T-gate, a gate outside the Clifford gates, makes from the stabilizer state $|+\rangle$. At the T-state the relation along an edge gives $M = ½$ at $\delta = (1 - 1/\sqrt{2})/4 \approx 0.0732$. Near a stabilizer state the magic grows as the square root of the infidelity, $M \approx \sqrt{(8\delta)}/\ln 2$. So the magic and the infidelity obey no linear relation. In general, a given δ leaves the magic free to lie anywhere in a band between two curves. The two curves are set by how the Bloch vector shares its length among its components.

I. The Two Measures

A qubit is a quantum system with two alternatives. A gate is one operation on one or two qubits. A Clifford circuit is a circuit built from three gates, the Clifford gates: the Hadamard gate and the phase gate, which act on one qubit, and the controlled-NOT, which acts on two qubits. The states that Clifford circuits prepare are called stabilizer states. A measurement in the computational basis reads each qubit as 0 or 1. The theorem of the physicists Gottesman and Knill establishes that a classical computer simulates efficiently every Clifford circuit that acts on stabilizer states and ends with measurements in the computational basis (Gottesman 1998; *The Quantum Bridge*, Chapter 4, Section VII). An efficient simulation needs an amount of work that grows no faster than a fixed power of the number of qubits and gates. The simulation is efficient although the circuits create superposition and entanglement. Superposition is the combining of alternatives into one state. Entanglement is the joining of several qubits into one state that cannot be written as a separate state for each qubit. This book belongs to a collection of books in which I give one account of physics. Another book of the collection sets out the stabilizer states and the magic states in full: *The Quantum Bridge; Entanglement, Error Correction, and the Limits of Simulation*.

Universal quantum computation is quantum computation that can carry out any operation on its qubits. Universal quantum computation needs a further resource that Clifford circuits cannot supply. Quantum computers usually get the further resource from T-gates or injected magic states. A T-gate is a gate outside the Clifford gates. Section II describes the T-gate as a rotation. An injected magic state is a state outside the stabilizer states, prepared apart from the circuit and then fed into the circuit. Magic monotones measure a state's magic, which is the amount of the further resource that the state carries. A magic monotone is a quantity that is zero on stabilizer states and does not increase under stabilizer operations, which are the operations a Clifford circuit can carry out: Clifford gates, the preparation of stabilizer states, and measurements in the computational basis.

Log-robustness. Let ρ stand for a state of the system, which may be pure or mixed: a pure state is a state described as completely as quantum mechanics allows, and a mixed state is a weighted mixture of pure states. The robustness of magic of a state ρ is the least total weight $\Sigma|x_i|$ over all ways of writing $\rho = \Sigma\, x_i\sigma_i$ with σ_i pure stabilizer states and x_i real numbers (Howard and Campbell 2017). The physicists Howard and Campbell defined the robustness of magic. The combination is in general not a mixture, because some of the weights x_i may be negative. The log-robustness of magic is the logarithm of the robustness, $M(\rho) = \log_2 R(\rho)$. The log-robustness is zero exactly on the mixtures of stabilizer states. The log-robustness does not increase under stabilizer operations, so the log-robustness is a magic monotone. The log-robustness is also subadditive: $M(\rho \otimes \sigma) \leq M(\rho) + M(\sigma)$, where $\rho \otimes \sigma$ is the joint state of two systems, with the first system in the state ρ and the second in the state σ. So the log-robustness of two systems together is at most the sum of the log-robustnesses of the two systems taken separately. In the literature the name "mana" belongs to a different measure (Veitch, Mousavian, Gottesman and Emerson 2014). Mana is built from the discrete Wigner function and defined for odd dimensions. The discrete Wigner function describes a state by a finite

grid of numbers, some of which may be negative. A system of odd dimension has an odd number of alternatives. A qubit has two alternatives, so for qubits this book uses the log-robustness.

Stabilizer infidelity. For a pure state ψ and a pure stabilizer state s, the overlap $|\langle s|\psi\rangle|^2$ is the probability that a test for the state s passes on the state ψ. The stabilizer fidelity of a pure state ψ is the largest overlap of ψ with a pure stabilizer state: $F_S(\psi) = \max |\langle s|\psi\rangle|^2$, the maximum over pure stabilizer states s. The stabilizer infidelity is $\delta(\psi) = \frac{1}{2}(1 - F_S(\psi))$. The stabilizer infidelity is zero exactly on stabilizer states. The stabilizer infidelity is unchanged by Clifford unitaries, the reversible operations that Clifford circuits carry out, because each Clifford unitary permutes the stabilizer states.

II. The Stabilizer Octahedron

A one-qubit state has a Bloch vector $r = (x, y, z)$, with $|r| = 1$ for pure states. The Bloch vector is an arrow that places the state in a ball of radius one. The pure states lie on the surface of the ball, which is called the Bloch sphere. One qubit has six pure stabilizer states: the eigenstates of the three Pauli operators, where an eigenstate of a Pauli operator is a state with a definite value along one of the three axes x, y and z. The six pure stabilizer states are the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$, where the three axes cross the Bloch sphere. The mixed stabilizer states are the mixtures of the six pure stabilizer states, so the Bloch vectors of the mixed stabilizer states fill the convex hull of the six points. The convex hull of a set of points is the smallest solid that contains every straight segment between two of the points. The convex hull of the six points is the octahedron $|x| + |y| + |z| \leq 1$, with six vertices, twelve edges and eight faces, which this book calls the stabilizer octahedron.

The Clifford unitaries act on the Bloch sphere as the rotations that carry the octahedron into itself. An overall phase is a single factor that multiplies the whole state and changes no measured quantity. Two unitar-

ies act on the Bloch sphere as the same rotation when they differ only by an overall phase. Taken modulo phases (with two unitaries that differ only by a phase counted as one), the Clifford unitaries of one qubit form the octahedral rotation group, of order 24. The order of a group is the number of its elements. The order 24 is what equation (4.10) of *The Quantum Bridge* gives for one qubit. The group SU(2) is a group of one-qubit unitaries in which each rotation of the Bloch sphere comes from exactly two elements. In SU(2), the preimage of the octahedral rotation group is the set of the elements that give the octahedral rotations. The preimage is the binary octahedral group, of order 48. A Clifford unitary permutes the six stabilizer states, while a non-Clifford rotation carries some stabilizer states off the vertices. The T-gate is one non-Clifford rotation: a rotation by $\pi/4$ about the z-axis.

The Incommensuration Theorem states that symmetry and continuity cannot both be perfected in any comparison. The Clifford action is the action of the Clifford unitaries on the states. The Clifford action is a permutation of a finite set of states, exact and discrete. The Clifford action is exact because it carries the six stabilizer states exactly onto one another. The Clifford action is discrete because it moves the states only among six separate points. In the terms of the Incommensuration Theorem, the Clifford action has the discontinuously symmetric character. The full rotation group is the group of all rotations of the Bloch sphere. The full rotation group reaches every point of the continuous sphere. In the terms of the Incommensuration Theorem, the full rotation group has the continuous character. A state has magic when the state lies off the finite set of states that the Clifford action preserves. *The Quantum Bridge* (Chapter 4, Section VII) identifies a computation confined to the stabilizer states as discontinuously symmetric in what the computation reaches. *The Quantum Bridge* also identifies the non-stabilizer resource as what lets a finite gate set reach the continuum of the formalism, the continuous range of states that the mathematics of quantum mechanics allows.

III. Both Measures from the Bloch Vector

Proposition 3.1 computes both measures from the Bloch vector. The ℓ^1 norm of a vector is the sum of the absolute values of its components, so the stabilizer octahedron is the set of points whose ℓ^1 norm is at most 1.

> **Proposition 3.1.** For a one-qubit state with Bloch vector r lying outside the octahedron, $R = |x| + |y| + |z|$, the ℓ^1 norm of r. For a pure state, $F_S = (1 + m)/2$, where $m = \max(|x|, |y|, |z|)$.

The robustness. Let $L = |x| + |y| + |z| > 1$ be the ℓ^1 norm of a Bloch vector r outside the octahedron. The point $-r/L$ has ℓ^1 norm 1, so the point $-r/L$ lies on the surface of the octahedron, as the Bloch vector of a mixed stabilizer state σ_-. Set $p = (L - 1)/2$. The point $r(1 - p/L)/(1 + p)$ has ℓ^1 norm $(L - p)/(1 + p)$, which equals 1, so this point also lies on the surface of the octahedron, as the Bloch vector of a mixed stabilizer state σ_+. The state is then the combination $\rho = (1 + p)\sigma_+ - p\sigma_-$, whose total weight is $1 + 2p = L$. Each mixed stabilizer state is a mixture of pure stabilizer states with positive weights that add to one, so writing σ_+ and σ_- as such mixtures leaves the total weight at L. No decomposition has a smaller total weight, because the Bloch vector of each stabilizer state has ℓ^1 norm at most 1, so the ℓ^1 norm of the Bloch vector of any combination $\Sigma\, x_i\sigma_i$ is at most $\Sigma|x_i|$. The Bloch vector of ρ has ℓ^1 norm L, so every decomposition of ρ has total weight at least L. So $R = L$. The T-state is $T|+\rangle$, the state that the T-gate makes from the stabilizer state $|+\rangle$, whose Bloch vector is $(1, 0, 0)$. For the T-state $T|+\rangle$, with $r = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $R = \sqrt{2}$ and $M = ½$. Howard and Campbell (2017) give the same value, $M = ½$, for the T-state.

The fidelity. For a pure state and a stabilizer state with Bloch vector s, $|\langle s|\psi\rangle|^2 = (1 + r \cdot s)/2$, where $r \cdot s$ is the dot product of the two Bloch vectors. Each of the six vertices has one component equal to ± 1 and the other two components equal to 0. So the largest $r \cdot s$ over the six vertices is m, the largest absolute component of r. Hence $F_S = (1 + m)/2$, as Proposition 3.1 states. For the T-state, $F_S = (1 + 1/\sqrt{2})/2 = \cos^2(\pi/8) \approx 0.8536$, so $\delta = (1 - 1/\sqrt{2})/4 \approx 0.0732$.

IV. The Relation Along an Edge

Rotate the stabilizer state $|+\rangle$, with Bloch vector $(1, 0, 0)$, about the z-axis by an angle θ between 0 and $\pi/4$. The Bloch vector of the rotated state is $(\cos\theta, \sin\theta, 0)$. The state $|+i\rangle$ is the stabilizer state with Bloch vector $(0, 1, 0)$. As θ runs from 0 to $\pi/4$, the rotated state traces an arc of the great circle above the edge from $|+\rangle$ to $|+i\rangle$, where a great circle is a circle on the Bloch sphere whose center is the center of the ball. For θ between 0 and $\pi/4$ the largest component is $\cos\theta$. So Proposition 3.1 gives the two measures on the arc, the magic M and the stabilizer infidelity δ:

$$M = \log_2(\cos\theta + \sin\theta), \qquad \delta = (1 - \cos\theta)/4 \qquad (3.1)$$

The second formula of (3.1) gives $\cos\theta = 1 - 4\delta$. So $\sin\theta = \sqrt{(8\delta(1 - 2\delta))}$. When $\cos\theta$ is replaced by $1 - 4\delta$ and $\sin\theta$ by $\sqrt{(8\delta(1 - 2\delta))}$, the first formula of (3.1) gives M as a function of δ alone:

> **Proposition 3.2 (magic against infidelity along an edge).** $M = \log_2(1 - 4\delta + \sqrt{(8\delta(1 - 2\delta))}), \quad 0 \le \delta \le (1 - 1/\sqrt{2})/4.$

At $\theta = \pi/4$ the rotated state is the T-state, for which Proposition 3.2 gives M = ½ and $\delta \approx 0.0732$. Every image of the arc under a Clifford unitary obeys the relation of Proposition 3.2, since both measures are unchanged by Clifford unitaries. The T-state is the point of greatest magic on the arc. The T-state lies above the midpoint of an edge of the octahedron.

Near the stabilizer state $|+\rangle$, the angle θ is small, so $M \approx \theta/\ln 2$ and $\delta \approx \theta^2/8$. Together the approximations $M \approx \theta/\ln 2$ and $\delta \approx \theta^2/8$ give

$$M \approx \sqrt{(8\delta)} \,/\, \ln 2 \qquad (3.2)$$

The magic grows as the square root of the infidelity. Magic is first order in the angle of departure from a stabilizer state: the magic grows in proportion to the angle θ. Infidelity is second order: the infidelity grows in proportion to the square of θ. So no relation $M = k\delta$ for a constant k holds near the stabilizer states, and so no single constant relates the magic to the infidelity along the arc. At the T-state the ratio M/δ is about 6.83, but the ratio grows without bound as the state approaches a stabilizer state.

V. The Band at Given Infidelity

For pure states off the arcs above the edges, neither measure determines the other, because R depends on all three components of r while F_S depends only on the largest. The infidelity of a pure state is $\delta = (1 - m)/4$, because $F_S = (1 + m)/2$, so the largest absolute component is $m = 1 - 4\delta$. For a pure state $|r| = 1$, so the two smaller components have squares summing to $1 - m^2$, while neither of the two smaller components exceeds m in absolute value. The two smaller components have the largest sum of absolute values when they are equal, while they have the smallest sum when one of them is zero. For $m \geq 1/\sqrt{2}$ both cases respect the condition that neither smaller component exceeds m, so both cases occur among pure states. The condition $m \geq 1/\sqrt{2}$ is the condition $\delta \leq (1 - 1/\sqrt{2})/4$ of Proposition 3.3.

> **Proposition 3.3 (the band of magic at given infidelity).** For a pure one-qubit state with stabilizer infidelity $\delta \leq (1 - 1/\sqrt{2})/4$ and $m = 1 - 4\delta$, $\log_2(m + \sqrt{(1 - m^2)}) \leq M \leq \log_2(m + \sqrt{(2(1 - m^2))})$.

The lower curve of Proposition 3.3 is the edge relation of Proposition 3.2, reached when one of the two smaller components is zero. The upper curve of Proposition 3.3 is reached by states on the great circles through a vertex and the center of a face, where the two smaller components are equal. Of all one-qubit pure states, the states above the face centers have the largest magic: for the face-center state $r = (1, 1, 1)/\sqrt{3}$, $R = \sqrt{3}$, $M = \log_2\sqrt{3} \approx 0.79$ and $\delta = (1 - 1/\sqrt{3})/4 \approx 0.106$. Published papers name the two kinds of extremal state, the edge state and the face-center state, differently from this chapter. The state $T|+\rangle$ is Clifford-equivalent to the edge state that the physicists Bravyi and Kitaev (2005) call $|H\rangle$. Two states are Clifford-equivalent when a Clifford unitary carries one of the two states into the other. The state that Bravyi and Kitaev call $|T\rangle$ is the face-center state.

VI. More Than One Qubit

The Quantum Passage is the book of this collection that derives quantum mechanics from the structure of comparison. The algebra that *The Quantum Passage* derives is called the comparison algebra. The comparison algebra has three parts, called carriers. In the comparison algebra a qubit is the space $\mathbb{C}^2$ on which the immanent carrier $\mathbb{H}$, the quaternions, acts. The quaternionic sector is the part of the comparison algebra that $\mathbb{H}$ carries. For n qubits the algebra of the quaternionic sector is $\mathbb{H}^{\otimes n}$, the tensor product of n copies of $\mathbb{H}$. A real algebra is an algebra whose numbers are the real numbers. As a real algebra, $\mathbb{H}^{\otimes n}$ is $M_{2^n}(\mathbb{R})$ for even n and $M_{2^{n-1}}(\mathbb{H})$ for odd n. The symbol $M_{2^n}(\mathbb{R})$ stands for the real square matrices of size 2^n. The symbol $M_{2^{n-1}}(\mathbb{H})$ stands for the quaternion square matrices of size 2^{n-1}. Over $\mathbb{C}$, with complex numbers allowed as coefficients, $\mathbb{H}^{\otimes n}$ generates $M_{2^n}(\mathbb{C})$, the complex square matrices of size 2^n, which act on the space of the n qubits. The rays of the n-qubit space form $\mathbb{C}P^{2^n-1}$, one ray for each pure state of the n qubits. The number of

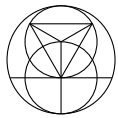
pure n-qubit stabilizer states is $2^n \prod_{k=1}^{n} (2^k + 1)$. The researchers in quantum computing Aaronson and Gottesman published the count, which is equation (4.11) of *The Quantum Bridge* (Aaronson and Gottesman 2004). The count is 6, 60 and 1,080 for one, two and three qubits.

Both measures extend to states of more than one qubit. The log-robustness is subadditive, so $M(\psi_1 \otimes \ldots \otimes \psi_n) \leq \Sigma\, M(\psi_k)$. The stabilizer fidelity is multiplicative for tensor products of one-qubit states, which means that the stabilizer fidelity of the product is the product of the separate stabilizer fidelities (Bravyi, Browne, Calpin, Campbell, Gosset and Howard 2019). So a tensor product of one-qubit states has infidelity $\frac{1}{2}(1 - \prod F_S(\psi_k))$. For n copies of the T-state, $M \leq n/2$ and $\delta = \frac{1}{2}(1 - \cos^{2n}(\pi/8))$, which approaches ½ as n grows, because $\cos^2(\pi/8)$ is less than one.

Both measures are unchanged by Clifford unitaries, so when one T-gate and then Clifford gates act on stabilizer states, the state they prepare has the magic and the infidelity of the single T-state, however many qubits the Clifford gates entangle (Chapter 4, Section III).

VII. Summary

For one qubit, the magic and the stabilizer infidelity are both read from the Bloch vector: the robustness is the ℓ^1 norm of the Bloch vector, and the stabilizer fidelity is one half of one plus the largest absolute component of the Bloch vector. Above the edges of the stabilizer octahedron, each of the two measures determines the other exactly (Proposition 3.2). The T-state has $M = \frac{1}{2}$ and $\delta \approx 0.0732$. In general, at a given infidelity, the magic lies in a band (Proposition 3.3). Near the stabilizer states the magic grows as the square root of the infidelity, so the magic and the infidelity obey no linear relation. The relation between the two measures is set by the geometry of the stabilizer octahedron alone.



Chapter 4

Entanglement, Magic and the Count of Non-Clifford Gates

Why Entanglement Distributes Magic and Does Not Create It

ABSTRACT

Clifford gates are the Hadamard gate, the phase gate and the controlled-NOT. Clifford gates can entangle any number of qubits to any extent, and Clifford gates add no magic. Magic is the further resource, beyond what Clifford circuits supply, that universal quantum computation requires. Clifford circuits prepare the stabilizer states, and the stabilizer states have no magic. So when a state is prepared from stabilizer states, its magic is at most the magic that the non-Clifford gates of the preparation bring in. In terms of the log-robustness M of Chapter 3, a circuit with t T-gates prepares only states with $M \leq t/2$, so any preparation of a state of magic M needs at least 2M T-gates. A T-state is the state $T|+\rangle$ that one T-gate makes from the stabilizer state $|+\rangle$. One T-gate followed by Clifford gates can spread the magic of a single T-state over any number n of qubits. An example is the n-qubit state $(|0...0\rangle + e^{i\pi/4}|1...1\rangle)/\sqrt{2}$. For every n, the n-qubit state has the magic and the stabilizer infidelity of one T-state: $M = \frac{1}{2}$ and $\delta \approx 0.0732$. Magic depends on the state alone, not on the preparation. So two preparations that end in one state give that state the same magic. Swaps of qubits are Clifford gates, and swaps can route every non-Clifford gate through one qubit, so for every state that is not a stabilizer state the least number of qubits that must receive non-Clifford gates is 1.

> The quantity that magic constrains is therefore the count of the non-Clifford gates themselves, not the count of the qubits the non-Clifford gates act on.

I. Entanglement Is Free

Entanglement is free in one exact sense: Clifford gates, which add no magic, can create entanglement of any extent. The Clifford group is the set of gates built from Clifford gates. The Clifford group contains two gates that act on two qubits at once and can entangle the two qubits: the controlled-NOT and the controlled-Z. A stabilizer circuit is a circuit of Clifford gates acting on stabilizer states. So stabilizer circuits can create entanglement of any extent. The theorem of Gottesman and Knill states that a classical computer can still simulate every stabilizer circuit efficiently (*The Quantum Bridge*, Chapter 4, Section VII). The cluster state of measurement-based computation is the clearest case of a state with much entanglement and no magic (Raussendorf and Briegel 2001). In measurement-based computation, a computer first prepares one large entangled state, the cluster state. The computer then computes by measuring single qubits of the cluster state (Section VI). The cluster state is a stabilizer state. Controlled-Z gates, which are Clifford gates, prepare the cluster state from qubits in the stabilizer state $|+\rangle$. So the cluster state, like every stabilizer state, has $M = 0$, however much entanglement the cluster state carries.

Entanglement is still needed, and Jozsa and Linden (2003) showed why. A pure state is a state known completely, with no classical uncertainty about which state is present. Jozsa and Linden considered a computation on pure states whose state, at every step, is a product of separate blocks of qubits, and Jozsa and Linden showed that when no block is larger than some fixed size, a classical computer can simulate the computation efficiently. In a product of separate blocks, entanglement joins qubits only within a block, so a pure-state computation that is to escape efficient classical simulation must build entanglement whose extent grows with

the size of the problem that the computation solves. Entanglement is therefore necessary for a pure-state computation to escape efficient classical simulation, but entanglement is not sufficient for the escape from efficient classical simulation, by the theorem of Gottesman and Knill. Stabilizer circuits create entanglement of any extent, and a classical computer still simulates stabilizer circuits efficiently. The rest of this chapter shows how the theorem of Jozsa and Linden and the theorem of Gottesman and Knill fit together: the non-Clifford gates supply the magic, and entanglement carries the magic across the qubits of a computation.

II. Entanglement Does Not Create Magic

The bound of this section (Proposition 4.1 below) follows from two facts. First, the log-robustness does not increase under stabilizer operations. The stabilizer operations are Clifford gates, Pauli measurements (measurements of the Pauli operators of Chapter 3), the addition of qubits in stabilizer states, and Clifford gates chosen by the outcomes of earlier measurements. Second, a Clifford circuit can apply a T-gate by gate injection, in which the circuit makes a Pauli measurement and then a Clifford correction, and consumes one copy of the magic state $T|+\rangle$ (*The Quantum Bridge*, Chapter 4, Section VII). Take a circuit of Clifford gates and t T-gates acting on stabilizer states, and replace each T-gate by gate injection. The circuit then becomes a stabilizer operation acting on stabilizer states together with t copies of $T|+\rangle$. The log-robustness is the logarithm of the robustness R. The robustness is submultiplicative: the robustness of two states taken together is at most the product of their robustnesses, $R(\rho \otimes \sigma) \leq R(\rho)R(\sigma)$. The robustness of the T-state is $R(T|+\rangle) = \sqrt{2}$ (Chapter 3, Proposition 3.1). A stabilizer state has robustness 1, since the log-robustness of a stabilizer state is zero. The input to the stabilizer operation is the stabilizer states together with the t copies of $T|+\rangle$. So the input has robust-

ness at most $(\sqrt{2})^t$, which is a log-robustness of at most $t/2$. The stabilizer operation cannot raise the log-robustness of the input. So Proposition 4.1 holds.

> **Proposition 4.1 (no amplification).** A state prepared from stabilizer states by Clifford gates and t T-gates has $M \leq t/2$. Equivalently, any such preparation of a state of log-robustness M uses at least 2M T-gates.

Howard and Campbell (2017) defined the robustness of magic (Chapter 3). Howard and Campbell used the robustness in the same way as Proposition 4.1: as a lower bound on the count of T-gates a preparation needs. The Clifford gates of a circuit can create any amount of entanglement, and the magic of the prepared state is still at most the magic that the T-gates bring in.

III. Entanglement Distributes Magic

Prepare the two-qubit state of equation (4.1) with one T-gate and one controlled-NOT.

$$|\psi\rangle = \text{CNOT} \cdot (\text{T} \otimes \text{I}) \cdot |+\rangle|0\rangle = (|00\rangle + e^{i\pi/4}|11\rangle)/\sqrt{2} \qquad (4.1)$$

Equation (4.1) is read from right to left. The first qubit of $|\psi\rangle$ starts in the stabilizer state $|+\rangle$, and the second qubit starts in the stabilizer state $|0\rangle$. The first qubit receives the T-gate, and the second qubit receives the identity gate I, which leaves the second qubit unchanged. Then the controlled-NOT acts on both qubits. One T-gate acts, and the controlled-NOT entangles the two qubits. Both the log-robustness and the stabilizer infidelity are unchanged by Clifford unitaries and by adding a qubit in a stabilizer state. So $|\psi\rangle$ has exactly the values of one T-state, $M = \frac{1}{2}$ and $\delta = (1 - 1/\sqrt{2})/4 \approx 0.0732$, because $|\psi\rangle$ is the T-state $\text{T}|+\rangle$ with a qubit added in the stabilizer state $|0\rangle$, followed by the controlled-NOT, which is a Clifford unitary. Prepare the n-qubit state $(|0\rangle^{\otimes n} + e^{i\pi/4}|1\rangle^{\otimes n})/\sqrt{2}$ with one T-gate and $n - 1$ controlled-NOT gates. For every number n of qubits, the n-

qubit state has the log-robustness and the stabilizer infidelity of one T-state. The magic of the n-qubit state is spread over all n qubits, while each qubit taken alone has no magic, because each qubit taken alone is in an equal mixture of $|0\rangle$ and $|1\rangle$, which is a mixture of stabilizer states.

The converse of spreading magic is concentrating magic, and physicists concentrate magic by magic state distillation. In magic state distillation, physicists turn many noisy copies of a magic state into fewer copies of higher purity, using Clifford gates, Pauli measurements and post-selection (Bravyi and Kitaev 2005). Post-selection means keeping a run only when the measurements give the wanted outcomes. Fifteen noisy copies yield one purer copy in the fifteen-to-one protocol of Bravyi and Kitaev (*The Quantum Bridge*, Chapter 4, Section VII). Clifford gates, Pauli measurements and post-selection cannot create magic, so the magic of each purer copy comes from the noisy copies consumed, and so distillation concentrates magic at a price: each purer copy is paid for with noisy copies.

IV. Magic Belongs to the State

The log-robustness and the stabilizer infidelity are functions of the state alone. Each is computed from the state, with no reference to how the state was made. So two preparations that end in the same state give that state the same values, whatever sequence of operations each preparation used. Take the T-state $T|+\rangle$, prepared in two ways. In the first way, a circuit applies a unitary T-gate to the stabilizer state $|+\rangle$. In the second way, a Clifford circuit applies the T-gate to $|+\rangle$ by gate injection, with a Pauli measurement and a Clifford correction. The circuit consumes a copy of $T|+\rangle$ held in an ancilla, which is an extra qubit used only in the preparation. The T-state prepared in either way has $M = \frac{1}{2}$ and $\delta \approx 0.0732$.

Suppose someone divides a prepared state's magic into two parts. One part is kept by the system that receives the state, and the other part is passed to an environment, the surroundings of the system. The sizes of

the two parts are set by the channel, the physical process that prepared the state, but the magic is a function of the state alone and does not depend on the channel. So the division is not a division of the magic. A record is a lasting trace in the surroundings. The two preparations of the T-state differ in what each preparation does to the surroundings: the unitary gate leaves no record, and the injection leaves a record, the outcome of the Pauli measurement. The magic of the prepared state is the same for both preparations. The cost of a preparation is the count of what the preparation consumes: the T-gates or the copies of $T|+\rangle$. Proposition 4.1 shows that a preparation of a state of log-robustness M consumes at least 2M T-gates or copies of $T|+\rangle$.

V. What Counts: Gates, Not Qubits

The magic of a state is not measured by the number of qubits that receive non-Clifford gates. A swap is the Clifford gate that exchanges the states of two qubits. A T-gate on qubit k is the same operation as three steps in a row: a swap of qubits 1 and k, a T-gate on qubit 1, and the same swap again. So swaps alone can move all the non-Clifford gates of any circuit onto one qubit. A state that is not a stabilizer state needs at least one non-Clifford gate, because Clifford gates alone prepare only stabilizer states. So for every state that is not a stabilizer state, 1 is the least number of qubits that must receive non-Clifford gates. The quantity that magic constrains is the count of the non-Clifford gates themselves, and Proposition 4.1 sets a lower bound of 2M on that count.

Three examples show the bound of Proposition 4.1 in use. First, take n copies of the T-state. The log-robustness is subadditive: the log-robustness of states taken together is at most the sum of their log-robustnesses, and each T-state has $M = \frac{1}{2}$. So the n copies have $M \leq n/2$. The bound says that at least 2M T-gates are needed, and 2M is at most n, so the n T-gates that prepare the n copies, one T-gate for each copy, satisfy the bound. Second, the n-qubit state of Section III spreads the magic of one T-state

over n qubits and has $M = \frac{1}{2}$. So the bound requires at least one T-gate, and one T-gate suffices. Third, when a Clifford unitary acts on n qubits, t of them in the T-state and the other $n - t$ in stabilizer states, the resulting state has $M \leq t/2$, however entangled the resulting state is.

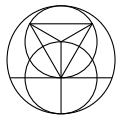
VI. Measurement-Based Computation

In measurement-based computation, the resource state is the cluster state, the entangled state that the computer prepares before any measurement. The cluster state has no magic, because the cluster state is a stabilizer state (Section I). A Pauli basis is the pair of stabilizer states that belong to one Pauli operator. In a measurement-based computation, the magic comes from measurements of single qubits in bases that are not Pauli bases, so the count of those measurements takes the place that the count of T-gates has in a circuit. The entanglement of the cluster state carries the effect of the measurements across the computation.

VII. Summary

Clifford gates create entanglement and add no magic. When a state is prepared from stabilizer states with t T-gates, the magic of the state is bounded by the T-gates used, $M \leq t/2$, so at least 2M T-gates are needed to prepare a state of log-robustness M (Proposition 4.1). Entanglement made by Clifford gates spreads the magic of one T-gate over any number of qubits without changing the amount of the magic, while physicists concentrate magic by distillation at a price: each purer copy is paid for with noisy copies. Entanglement is necessary for a pure-state computation to escape efficient classical simulation, but entanglement is not sufficient: the non-Clifford gates supply the magic, and entanglement spreads the magic over the qubits. Magic is a function of the state alone, so two preparations

of one state give that state the same magic. The count that magic constrains is the count of non-Clifford gates, not the count of the qubits that those gates act on.



Chapter 5

The Three Units of c_1

Why Each of the Three Directions Weighs the Same

ABSTRACT

The transcendent carrier $M_3(\mathbb{C})$ is the algebra of 3×3 complex matrices, which act on the module $\mathbb{C}^3$, the space of columns of three complex numbers. The integer $c_1 = 3$ is the rank of the transcendent carrier. The rank is the number of independent directions in which the content of the module $\mathbb{C}^3$ can vary. Re-projection, a postulate of the collection, sets the number of directions equal to the number of modalities. The three directions of $\mathbb{C}^3$ have equal weight, because the symmetries of the carrier carry each direction into each other direction. The symmetries of $M_3(\mathbb{C})$ form its unitary group, and the unitary group permutes any orthonormal basis of $\mathbb{C}^3$ and moves every unit vector to every other unit vector. Take any quantity that is assigned direction by direction and respects the symmetries of the carrier. Each direction gets an equal share of the quantity: one third of the whole. The omniscient, immanent and transcendent modalities have distinct roles. But the distinct roles do not make the shares unequal, because a difference of role is not a difference of size. The pure states of the transcendent carrier form the space $\mathbb{C}P^2$, and the topology of $\mathbb{C}P^2$ agrees: the Euler characteristic of $\mathbb{C}P^2$ is 3, with one independent class in each even degree. So c_1 and $\chi(\mathbb{C}P^2)$ are the same count: $c_1 = \chi(\mathbb{C}P^2) = 3$. The count c_1 enters one relation of the collection's account of

physics: the vacuum relation $\rho_\Lambda = c_1 \, f \, M_{Pl}{}^4/S$, which sets the energy density of empty space. In the vacuum relation, c_1 is read on the context instance of the internal space, the space of possibility of *The Gravitational Passage*. On the context instance, the factor $\mathbb{C}P^2$ is the space of the pure states of a system with three alternatives. The argument from symmetry applies to the three alternatives just as the argument applies to the three directions of $\mathbb{C}^3$. So the three units of c_1 are three equal terms.

I. What c_1 Is

The transcendent integer is defined in Definition 11.1 of *Non-Commutative Geometry and the Immanent Metaphysics*, another book of the collection. Definition 11.1 reads $c_1 = n$, "the rank of the non-division carrier $M_n(\mathbb{C})$, which is also the coefficient of the first Chern class of its pure state space, $c_1(T\mathbb{C}P^{n-1}) = n \cdot h$ with h the hyperplane class". $M_n(\mathbb{C})$ is the algebra of $n \times n$ complex matrices. $M_n(\mathbb{C})$ is called a non-division carrier because some of its nonzero matrices have no inverse. The first Chern class is a topological quantity of a complex space. On the space of the pure states of $M_n(\mathbb{C})$, the first Chern class is n times the hyperplane class h. Section IV describes the hyperplane class. Definition 11.1 also says how the integer measures its carrier: "the transcendent by the dimension of its module, the number of independent directions in which its content can vary". The transcendent carrier $M_3(\mathbb{C})$ has the module $\mathbb{C}^3$. The 3×3 matrices of $M_3(\mathbb{C})$ act on the columns of $\mathbb{C}^3$ by multiplication.

Re-projection is Postulate P-Rep(i) of *Non-Commutative Geometry and the Immanent Metaphysics*, and re-projection fixes $n = 3$. Re-projection says: "The internal degrees of freedom of the content repeat the structure of comparison", so that "the complex rank n of the module of the transcendent carrier equals the number of modalities". There are three modalities, the omniscient, the immanent and the transcendent, so by re-projection n

= 3. The space $\mathbb{C}P^2$ is the space of the pure states of $M_3(\mathbb{C})$. The Euler characteristic of $\mathbb{C}P^2$ is also 3: $\chi(\mathbb{C}P^2) = 3$. The Euler characteristic is a topological count of the shape of a space, and Section IV works out the count for $\mathbb{C}P^2$.

The integer c_1 counts the directions of one carrier, $M_3(\mathbb{C})$. Re-projection sets the number of directions of $M_3(\mathbb{C})$ equal to the number of modalities. Re-projection fixes a count of directions. Re-projection does not attach a modality to any one direction, and Section II shows that no structure of the carrier distinguishes one direction from another.

II. Equal Weight from the Symmetries of the Carrier

The unitaries of $M_3(\mathbb{C})$ are the matrices that keep every length and every angle in $\mathbb{C}^3$ unchanged. The unitaries of $M_3(\mathbb{C})$ form the group $U(3)$. The unitaries act on $\mathbb{C}^3$ by matrix multiplication. An orthonormal basis of $\mathbb{C}^3$ is a set of three vectors of length one, each at right angles to the other two. A permutation matrix reorders the vectors of a basis, for example by exchanging the first vector with the second. Every permutation matrix is unitary, so $U(3)$ permutes any orthonormal basis of $\mathbb{C}^3$. $U(3)$ does more than permute a basis: $U(3)$ carries any orthonormal basis to any other orthonormal basis, and $U(3)$ carries any unit vector (a vector of length one) to any other unit vector. So no direction of $\mathbb{C}^3$ differs from another direction in any property that the unitaries of $M_3(\mathbb{C})$ leave unchanged.

> **Proposition 5.1 (equal weight).** Let a quantity Q be assigned to the directions of an orthonormal basis of $\mathbb{C}^3$, additively, and unchanged by the unitaries of $M_3(\mathbb{C})$. Then each direction carries Q/3.

Proposition 5.1 says that when a quantity Q is shared out among the three directions of a basis, and the unitaries leave the quantity unchanged, each direction carries one third of Q. The permutations of the basis are among the unitaries, and the permutations carry each direction to each other direction, so the three shares are equal. The three shares sum to Q, because Q is assigned additively, and so each share is Q/3.

The Standard Model is the theory of the known particles and of every force other than gravity. In the Standard Model, color symmetry is the symmetry that carries each direction of $\mathbb{C}^3$ to each other direction. The space $\mathbb{C}^3$ is the space of the three colors that a quark can carry (Chapter 1, Section II). The color symmetry is exact. The exactness of the color symmetry is why no measurement distinguishes one color from another.

III. Distinct Roles, Equal Weight

The three modalities are the omniscient, the immanent and the transcendent. Axiom III states that the three modalities are distinct, inseparable and non-interchangeable (*The Quantum Passage*, Chapter 4, Section IV). Non-interchangeability constrains the role that each modality plays. Non-interchangeability does not constrain the size of any share. The roles of the three modalities differ in kind, and a system in which two of the roles are exchanged does not hold together. Proposition 5.1 is about a size, a quantity assigned to the three directions of $\mathbb{C}^3$, and Proposition 5.1 fixes the shares of the size through the symmetry of the carrier. A difference of role is not a difference of size. So by Axiom III the roles of the modalities are distinct, and the three shares are still equal. *The Gravitational Passage* makes the point of this section, that distinct roles leave the shares equal, for the three channels that c_1 counts in the vacuum relation (Section V of this chapter sets out the vacuum relation). *The Gravitational Passage* says (Chapter 13, Section II): "The three members of the content row stand alike, each as what is registered."

IV. The Topology Agrees

The space $\mathbb{C}P^2$ is the complex projective plane, the ray space of $\mathbb{C}^3$: each point of $\mathbb{C}P^2$ is one ray, a complex line through the origin of $\mathbb{C}^3$. Each ray is one pure state of the transcendent carrier. The space $\mathbb{C}P^2$ has a cell structure with one cell in each of the real dimensions 0, 2 and 4. A cell structure is a way of building a space from cells: a point is a cell of dimension 0, a disk is a cell of dimension 2, and a solid ball of four dimensions is a cell of dimension 4. Each of the three cells of $\mathbb{C}P^2$ contributes +1 to the Euler characteristic, so $\chi(\mathbb{C}P^2) = 3$.

Betti numbers count the independent classes of a space in each degree. The Betti numbers of $\mathbb{C}P^2$ are $b_0 = b_2 = b_4 = 1$ and $b_1 = b_3 = 0$. The integral cohomology of a space is the algebra of the classes of the space, with whole-number coefficients. The mathematician Allen Hatcher works out the topology of $\mathbb{C}P^2$ in his textbook *Algebraic Topology*. The integral cohomology of $\mathbb{C}P^2$ is the ring $\mathbb{Z}[h]/(h^3)$, generated by the hyperplane class h, with 1, h and h^2 as the three generators of the ring (Hatcher 2002, Chapters 2 and 3). The cohomology ring of $\mathbb{C}P^2$ is made of the polynomials in h with whole-number coefficients, with h^3 set equal to zero. In $\mathbb{C}P^2$ a hyperplane is a complex projective line, and the hyperplane class h is the class of a complex projective line. The generators 1, h and h^2 each generate a part of the cohomology of rank one. A part of rank one is a single copy of the whole numbers. So the topology of $\mathbb{C}P^2$ has three equal units, as the symmetry of the carrier does.

The equality of the units is a property of every complex projective space. For every n, the Betti numbers of $\mathbb{C}P^n$ are 1 in each even degree from 0 to 2n and 0 in every other degree, so $\chi(\mathbb{C}P^n) = n + 1$ is a count of equal units. A four-dimensional space of another kind need not count equal units. A K3 surface is a four-dimensional space studied in complex

geometry. A K3 surface has $b_0 = b_4 = 1$ and $b_2 = 22$, so the middle degree holds 22 of the Euler characteristic 24 of a K3 surface. The equal count of this chapter belongs to $\mathbb{C}P^2$, the ray space of the transcendent carrier.

V. The One Relation That Carries c_1

The integer c_1 enters one relation of the collection's account of physics: the vacuum relation of the observer's particle horizon (*The Gravitational Passage; Deriving General Relativity from the Immanent Metaphysics*, Chapter 13, Section II; *The Quantum Bridge*, Chapter 1, Section V). The particle horizon of an observer is the boundary of the region from which light has had time to reach the observer. The vacuum relation is equation (5.1):

$$\rho_\Lambda = c_1 \, f \, {M_{Pl}}^4 / S, \qquad f = 1 - 3/(8\pi) = 0.881, \qquad S = \pi R^2 / {l_{Pl}}^2 \qquad (5.1)$$

In equation (5.1), ρ_Λ is the vacuum energy density that an observer inside the universe finds. M_{Pl} is the Planck mass, and l_{Pl} is the Planck length: the natural units of mass and length built from the constant of gravitation, the speed of light and Planck's constant. S is the Bekenstein-Hawking count. The Bekenstein-Hawking count is the number of degrees of freedom that the physicists Jacob Bekenstein and Stephen Hawking assigned to a black hole's horizon. *The Gravitational Passage* assigns the count S, by an identification, to the particle horizon of radius R.

The accessible share is the part of the vacuum energy that an observer inside the universe can find by measurement. *The Gravitational Passage* takes the accessible share of the external value $f \, {M_{Pl}}^4$ to be c_1/S, "one degree of freedom for each channel, among the S degrees of freedom the horizon counts". *The Gravitational Passage* counts the channels by $\chi(\mathbb{C}P^2)$ and writes the count as c_1, because on every complex projective space the Euler characteristic equals the coefficient c_1 of the first Chern class (Section I). *The Gravitational Passage* states why the Euler characteristic and the coefficient c_1 are equal: "each counts the alternatives of the system whose pure states the space is" (Chapter 13, Section II).

The Gravitational Passage counts the channels on a $\mathbb{C}P^2$. The $\mathbb{C}P^2$ of the channels is the first factor of the context instance of the internal space $\mathbb{C}P^2 \times S^2$. The context instance is the internal space of *The Gravitational Passage*, read as a space of possibility, with radii. *The Quantum Passage* reads the content instance and the context instance side by side as "two instances of one schema, read at their own scopes: on the side of content, as here, a space of states with no size; on the side of context, as there, a geometry with radii" (its Chapter 6, Section V). A schema is one mathematical form read in more than one way. In Sections I to IV of this chapter, $\mathbb{C}P^2$ is the space of pure states of the transcendent carrier. In Sections I to IV, $\mathbb{C}P^2$ belongs to the content instance: the schema read as a space of states with no size.

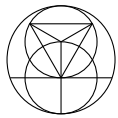
The Gravitational Passage reads the $\mathbb{C}P^2$ of the context instance as a space of pure states: "$\mathbb{C}P^2$ is also the space of the pure states of a quantum system with three alternatives" (Chapter 9, Section IV). So on the context instance c_1 counts three alternatives, one for each channel. For a system with three alternatives, the pure states are the rays of a three-dimensional complex space. The three alternatives are an orthonormal basis of the three-dimensional complex space. In the basis of the three alternatives, the unitaries of the three-dimensional complex space are the unitary 3×3 matrices. The unitary 3×3 matrices form the group $U(3)$, which acts on $\mathbb{C}^3$ as the unitaries of $M_3(\mathbb{C})$ act in Section II, and $U(3)$ carries each alternative to each other alternative. So Proposition 5.1 applies to the three alternatives, just as Proposition 5.1 applies to the three directions of $\mathbb{C}^3$ in Section II. Each channel takes an equal share: the share c_1/S is three equal terms $1/S$.

The equality of the three shares agrees with the third step of the argument of *The Gravitational Passage*, in which each channel engages one degree of freedom of the count S and the degrees of freedom hold equal parts. The symmetry of the space of possibility is a second ground for the equality of the three shares. The content instance and the context instance share three pieces of mathematics: the space $\mathbb{C}P^2$, the group $U(3)$ and

Proposition 5.1. But the two instances are two different spaces. The transcendent carrier acts on the content instance, and *The Gravitational Passage* counts the channels on the context instance.

VI. Summary

The integer $c_1 = 3$ counts the three directions of the module $\mathbb{C}^3$ of the transcendent carrier, and re-projection sets the count of directions equal to the number of modalities. The symmetries of the carrier carry each direction to each other direction. So each direction carries one third of any quantity that is assigned direction by direction and respects the symmetries of the carrier (Proposition 5.1). By Axiom III the roles of the modalities are distinct, and the shares stay equal. The topology of $\mathbb{C}P^2$ also has three equal units, one independent class in each even degree. In the vacuum relation, the one relation that carries c_1, c_1 counts the channels on the context instance. The $\mathbb{C}P^2$ of the context instance is the space of pure states of a system with three alternatives. Each alternative takes an equal share, by the symmetry of Proposition 5.1. So the share c_1/S is three equal terms $1/S$.



Chapter 6

The Born Rule and the Price of Being Inside

One Theorem at the Scale of a Measurement and at the Scale of the Horizon

ABSTRACT

The Born rule states the probability of each outcome of a single measurement. The Born rule is named for the physicist Max Born. *The Gravitational Passage* is the book of the collection that derives general relativity. The vacuum relation of *The Gravitational Passage* sets the cosmological constant (the energy density of empty space) from the share of the vacuum's energy that is accessible to an observer inside the vacuum. The accessible share is the share the observer can find by measuring. One step joins the Born rule and the vacuum relation: each is an instance of the Incommensuration Theorem. An instance of a theorem is a case to which the theorem applies. A lawful description is a description of a system by laws. The Incommensuration Theorem states that symmetry and continuity cannot both be perfected in any comparison. So every lawful description has a residual: a part of what is described that the description does not contain. *The Quantum Passage* is the book of the collection that derives quantum mechanics. *The Quantum Passage* gives the residual as the reason why a single outcome is a matter of probability. *The Gravitational Passage* identifies the information deficit of an observer inside the vacuum as an instance of the Incommensuration Theorem at the cosmological scale, which is the scale of the observer's horizon. The observer's horizon is the boundary of the region from which light has had time to reach the observer. The information deficit is the information about the vacuum that

the observer cannot access. The observer has the information deficit because the observer is part of the same system as the vacuum that the observer measures. The Born rule is the local price of comparing from inside, and the cosmological constant is the global price. A price of comparing from inside is a result that an observer inside a system gets in place of full knowledge of the system. The local scale is the scale of a single measurement. The global scale is the scale of the horizon. At a single measurement, the observer gets the weights of the outcomes from the Born rule in place of the outcome itself, while at the scale of the horizon the observer gets the cosmological constant in place of the whole vacuum energy.

I. The Vacuum Relation

An observer who measures the vacuum energy density is inside the universe whose vacuum the observer measures. The vacuum energy density is the energy in each unit of volume of empty space. *The Gravitational Passage* describes the observer's position (Chapter 13, Section I): "There is no laboratory outside the universe, and no control sample of empty space that the observer can compare to. The observer, the apparatus, and the vacuum are all parts of the same system." Degrees of freedom are the independent quantities that together describe a physical system. The observer's own degrees of freedom are excitations of the fields whose ground-state energy the observer measures, so the observer is made of the same fields as the vacuum. The ground state of a field is the lowest state of the field, and the ground-state energy is the energy of the field in the ground state. The ground-state energy of the fields is the vacuum energy. The observer's degrees of freedom are entangled with the vacuum: the observer and the vacuum share one quantum state, which cannot be written as a separate state for the observer and a separate state for the vacuum. So some of the information about the vacuum is held in the correlation

between the observer and the vacuum. The information held in the correlation is not accessible to the observer. The observer's information deficit is the information about the vacuum that is not accessible to the observer.

The Gravitational Passage sets the cosmological constant equal to the accessible share of the vacuum energy. The accessible share is the part of the vacuum energy that an observer inside the universe can find by measurement. *The Gravitational Passage* also obtains the vacuum relation for the observer's particle horizon (Chapter 13, Section II). The particle horizon of an observer is the boundary of the region from which light has had time to reach the observer. Chapter 5 of this book sets out the vacuum relation as equation (5.1), $\rho_\Lambda = c_1 \, f \, {M_{Pl}}^4/S$. In the vacuum relation, ρ_Λ is the vacuum energy density that an observer inside the universe finds. The symbol c_1 is the count of channels (Chapter 5). The symbol f is a numerical factor (Chapter 5). M_{Pl} is the Planck mass, the natural unit of mass built from the constant of gravitation, the speed of light and Planck's constant. S is the number of degrees of freedom of the horizon.

The Quantum Bridge, another book of the collection, works out the vacuum relation in numbers (Chapter 1, Section V, and Chapter 8, Section IV). For a particle horizon of radius 46.1 billion light years, $S = 2.29 \times 10^{123}$, so the vacuum relation gives a vacuum energy density of 2.56×10^{-47} GeV^4. The observed value is 2.52×10^{-47} GeV^4. GeV^4 is a unit of energy density used in particle physics. The vacuum relation is set out for general readers in Chapters 11 to 13 of another book of the collection, *The View from Inside; On the Cosmological Constant and Why Empty Space Weighs So Little*. *The View from Inside* uses the picture of a fish that sets out to measure the water the fish swims in (Chapter 11). The fish stands for the observer and the water for the vacuum, because the fish is inside the water that the fish measures, as the observer is inside the vacuum that the observer measures.

II. Two Instances of One Theorem

At the scale of a single measurement, *The Quantum Passage* gives the Incommensuration Theorem as the reason why the outcome of a measurement is a matter of probability (Chapter 7, Section II). The Incommensuration Theorem states that symmetry and continuity cannot both be perfected in any comparison. So every lawful description has a residual, a part of what is described that the description does not contain. *The Quantum Passage* says of the residual: "The single outcome of a quantum measurement is the plainest instance of the residual: the state before the measurement, known as fully as the theory allows, gives the weight of each outcome, and the outcome found is not contained in it." The state in the quotation is the measured system's quantum state. The weight of an outcome is the probability of the outcome. The Born rule determines the weight of each outcome from the state before the measurement (*The Quantum Passage*, Chapter 7).

At the scale of the horizon, *The Gravitational Passage* says of the observer inside the vacuum: "A system cannot fully measure itself. I identify this deficit as an instance of the Incommensuration Theorem at the cosmological scale." (*The Gravitational Passage*, Chapter 13, Section I.) The deficit in the quotation is the information deficit of Section I: the information about the vacuum that is held in the correlation between the observer and the vacuum, and that is not accessible to the observer.

So the Born rule and the vacuum relation both depend on one theorem, the Incommensuration Theorem, at the scale of a single measurement and at the scale of the horizon. The comparer is the observer who makes the measurement. In a single measurement, the quantum state that the comparer assigns cannot contain the outcome that the comparer will find, but the Born rule determines the weights of the outcomes that the comparer can find. In the measurement of the vacuum, the comparer is inside the vacuum, so the comparer cannot access the whole of the vacuum en-

ergy that the comparer measures, but the vacuum relation determines the share of the vacuum energy that is accessible to the comparer. The Born rule is the local price of comparing from inside, and the cosmological constant is the global price.

The two prices are computed by different steps. The Born weights are the weights that the Born rule determines. The Born weights come from Gleason's theorem on the derived Hilbert space (*The Quantum Passage*, Chapter 7, Sections III to V). Andrew Gleason, a mathematician, proved Gleason's theorem, which is about how probabilities can be assigned to the outcomes of measurements on a Hilbert space. A Hilbert space is the space of states of a quantum system. The derived Hilbert space is the Hilbert space that *The Quantum Passage* derives. The vacuum share is the accessible share of the vacuum energy. The vacuum share comes from two counts: the count of channels c_1 and the count S of the degrees of freedom of the horizon (*The Gravitational Passage*, Chapter 13, Section II). I join the Born rule and the vacuum relation through the Incommensuration Theorem, of which both are instances, but I do not derive either of the two numbers, the Born weights and the vacuum share, from the other.

III. Summary

The vacuum relation sets the cosmological constant from the share of the vacuum's energy that is accessible to an observer inside the vacuum: $\rho_\Lambda = c_1 \, f \, M_{Pl}{}^4/S$. The vacuum relation is carried by *The Gravitational Passage*, Chapter 13, and by *The View from Inside*, which means that each book sets out the vacuum relation. The Born rule and the vacuum relation are each an instance of the Incommensuration Theorem. The instance of the Born rule is the residual of a single measurement, whose outcome the state before the measurement does not contain. The instance of the vacuum relation is the deficit of an observer inside the vacuum that the observer measures. The Born rule is the local price of comparing from inside, and the cosmological constant is the global price.

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